Analysis III: Auditorium exercise class

Jacobian Matrix, Directional derivative, Vector Operators (curl/rot, div, grad), Taylor Polynomial

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BITTE BEACHTEN SIE DIE 3G-REGEL!
PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

- VOLLSTÄNDIG GEIMPFTE
- GENESENE
- GETESTETE

(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

- FULLY VACCINATED
- RECOVERED
- TESTED

(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

Vector Field

Let $D \subset \mathbb{R}^n$. The function $f : D \to \mathbb{R}^n$ is called a vector field on D.

If every function $f_i(x)$ of $f = (f1, ..., fn)^T$ is a C^k –function, then f is called C^k – vector field.

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

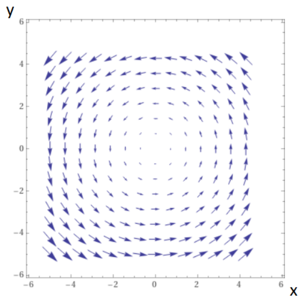

Figure 1: Sketch of the vector field $(-y, x)^T$.

Jacobian Matrix

Let $f : D \to \mathbb{R}^m, D \subset \mathbb{R}^n, x = (x_1, x_2 ..., x_n)^T \in D$,

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ ... \\ f_m(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1, ..., x_n) \\ f_2(x_1, ..., x_n) \\ ... \\ f_m(x_1, ... x_n) \end{pmatrix}$$

Then the Jacobian Matrix is $m \times n$ matrix $J_{ij} = \frac{\partial f_i}{\partial x_j}(x)$:

$$Jf(x) = \begin{pmatrix} \text{grad } f_1(x) \\ \text{grad } f_2(x) \\ ... \\ \text{grad } f_m(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & ... & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & ... & \frac{\partial f_2}{\partial x_n}(x) \\ ... & & & ... \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & ... & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

Jacobian determinant

- If $m = n$, the determinant of the Jacobian matrix is known as the Jacobian determinant of f.
- The Jacobian is used when making a change of variables and a coordinate transformation.

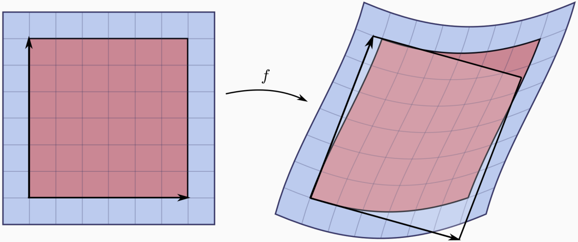

Figure 2: The Jacobian at a point gives the best linear approximation of the distorted parallelogram.[1]

[1] https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant

Exercise 1

Compute the Jacobian matrix and the Jacobian determinant of the following vector function:

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} x^2 y \\ 5x + \sin(y) \end{pmatrix}.$$

$Jf(x, y) =$

$\det(Jf(x, y)) =$

Exercise 2

Compute the Jacobian matrix of the following vector function:

$$f(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{pmatrix} = \begin{pmatrix} 2xy + yz^2 \\ e^{x^2+2y^2} \end{pmatrix}.$$

Chain rule for vector functions

- single-variable calculus

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$$

- multivariable calculus

$$J(f \circ g)(x) = Jf(g(x))Jg(x)$$

Exercise

Let $f : \mathbb{R}^3 \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}^3$ be a vector-valued function of one variable defined as follows

$$f(x, y, z) = e^z \cos(2x) \sin(3y),$$

$$g(t) = (x(t), y(t), z(t)) = (2t, t^2, t^3)$$

Compute the derivative of the composition $f \circ g$.

$\text{grad } f(x) = (f_x, f_y, f_z) =$

$g'(t) =$

Now apply Chain Rule:

$\frac{\partial f}{\partial t} = \text{grad } f(x(t), y(t), z(t)) \cdot g'(t) =$

Exercise

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^2y + xy^2,$$

and let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows

$$g(s, t) = \begin{pmatrix} x(s, t) \\ y(s, t) \end{pmatrix} = \begin{pmatrix} 2s + t \\ s - 2t \end{pmatrix}$$

Compute the gradient of the composition $f \circ g$.

Streamlines

Let $u = (u_1(x,y), u_2(x,y))^T$ be a velocity field of the two-dimensional flow. The streamlines associated with the flow u are the solutions of the system of differential equations

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \end{cases}$$

or the differential equation

$$y'(x) = \frac{v(x,y)}{u(x,y)}$$

(depending on parametrization).

Exercise

Calculate the streamline passing through a point $(x_0, y_0)^T$ for the stagnation point flow $u = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

Exercise

Calculate the streamline passing through a point $(x_0, y_0)^T$ for the time-dependent flow $u = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} U_0 \\ kt \end{pmatrix}$, where U_0 and k are constants.

Exercise

Calculate the streamline passing through a point $(0,0)^T$ for the flow

$$u = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} x \\ x(x-1)(y+1) \end{pmatrix}.$$

Directional Derivative

Let $f : D \to \mathbb{R}, D \subset \mathbb{R}^n$ - open, $x^0 \in D, v \in \mathbb{R} \backslash \{0\}$. Then

$$D_v f(x^0) := \lim_{t \to 0^+} \frac{f(x^0 + tv) - f(x^0)}{t}$$

is called the directional derivative of $f(x)$ in the direction v.

- theorem in the lecture $\implies$

$$D_v f(x^0) = \operatorname{grad} f(x^0) \cdot v$$

Exercise

Calculate by definition the directional derivative of the function $f(x_1, x_2) = 2x_1 + x_1x_2$ at a point (x_1^0, x_2^0) in the direction $v = (v_1, v_2)^T$.

Exercise

Let $f(x, y) = x^2y$. Compute

- grad $f(3, 2)$
- the derivative of f in the direction of $(1, 2)$ at the point $(3, 2)$

Exercise

Let $f(x, y) = x^2 y$. Compute

- grad $f(3, 2)$
- the derivative of f in the direction of $(2, 1)$ at the point $(3, 2)$

Exercise

Determine $D_v f(x,y)$ for $f(x,y) = cos(\frac{x}{y})$ in the direction $v = (3,-4)$.

Exercise

Determine $D_v f(3, -1, 0)$ for $f(x, y, z) = 4x - y^2 e^{3xz}$ in the direction $v = (-1, 4, 2)$. Is it a direction of descent or ascent?

Exercise

Determine $D_v f(3, -1, 0)$ for $f(x, y, z) = 4x - y^2 e^{3xz}$ in the direction $v = (-1, 4, 2)$. Is it a direction of descent or ascent?

Divergence

Let $f = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \ldots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}$ be a vector field.

The divergence of the vector field f is a scalar field defined as

$$\operatorname{div} f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}.$$

Rotation

Let $f = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$ be a three-dimensional vector field. The rotation of f (denoted rot f or curl f) is a vector field defined as

$$\operatorname{rot} f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$= \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix}$$

Exercise

Compute div f and rot f for $f(x, y, z) = \begin{pmatrix} x^2 y \\ -(z^3 - 3x) \\ 4y^2 \end{pmatrix}$

Exercise

Compute div f and rot f for $f(x, y, z) = (3x + 2z^2)\hat{i} + \frac{x^3y^2}{z}\hat{j} - (z - 7x)\hat{k}$

Level Surface (Isosurface)

- 3d analogue to level curves.

The equations of level surfaces are given by

$$f(x, y, z) = C, \forall C \in \mathbb{R}$$

i.e. the level surface equation at a point (x_0, y_0, z_0) is given by

$$N_{x_0} = \{x \in \mathbb{R}^3 : f(x, y, z) = f(x_0, y_0, z_0)\}$$

Taylor Polynomial

- in $\mathbb{R}$ for $f(x)$ around point x_0

$$T(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots$$

- in $\mathbb{R}^2$ for $f(x)$ around point (x_0, y_0)

$$T_1(x) = f(x_0, y_0) + \frac{f_x(x_0, y_0)}{1!}(x - x_0) + \frac{f_y(x_0, y_0)}{1!}(y - y_0)$$

- in $\mathbb{R}^3$ for $f(x)$ around point (x_0, y_0, z_0)

$$\begin{aligned} T_1(x) = f(x_0, y_0, z_0) &+ \frac{f_x(x_0, y_0, z_0)}{1!}(x - x_0) \\ &+ \frac{f_y(x_0, y_0, z_0)}{1!}(y - y_0) + \frac{f_z(x_0, y_0, z_0)}{1!}(z - z_0) \end{aligned}$$

Taylor Polynomial 2 Order

- in $\mathbb{R}$ for $f(x)$ around a point x_0

$$T_2(x) = T_1(x) + \frac{1}{2!}(x - x_0)f''(x_0)(x - x_0).$$

- in $\mathbb{R}^n$ for $f(\mathbf{x})$ around a point $\mathbf{x}_0$

$$T_2(\mathbf{x}) = T_1(\mathbf{x}) + \frac{1}{2!}(\mathbf{x} - \mathbf{x}_0)Hf(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$

Exercise: $T_2(x)$ for $f(x, y)$

Compute the second-degree Taylor polynomial of $f(x, y) = e^{-(x^2+y^2)}$ at a point $(0, 0)$.

$T_2(x)$ for $f(x, y, z)$

Compute the Jacobian matrix of the following vector function:

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \\ f_3(x, y) \end{pmatrix} = \begin{pmatrix} \sin(y) \\ x^3 + \cos(x) \\ x^2 y^2 \end{pmatrix}.$$

Exercise: $T_2(x)$ for $f(x, y, z)$

Compute the second-degree Taylor polynomial of $f(x, y, z) = \sin(x + y) + xe^{z-y} - z^2 + y$ at a point $x_0 = (0, 0, 0)$.

Thank you!