Classical and Quantum Lagrangian Field Theories on Manifolds with Boundary

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Outline

Introduction

- Generalize Segal–Atiyah's axioms to perturbative QFTs
  boundaries $\rightsquigarrow$ vector spaces
  manifolds (with boundaries) $\rightsquigarrow$ states/operators
- Do it for general Lagrangian theories (including gauge theories)
- First understand classical picture
- then the perturbative quantum BV picture

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- then the perturbative quantum BV picture

Lagrangian Mechanics

- In Lagrangian mechanics $S = \int_{t_0}^{t_1} L\, dt$ as a functional on the path space $N^{[t_0,t_1]}$.
- Usual example: $L = \frac{1}{2} m||v||^2 - V(q)$.
- Newton's equation are recovered as Euler–Lagrange equations (EL), i.e., critical points: $\delta S = 0$.
- A solution is uniquely specified by its initial conditions. Set $C := TN$, the space of **Cauchy data**.
- For this, one sets conditions at t_0 and t_1 (usually by fixing the path endpoints). Otherwise

$$\delta S = \mathrm{EL} + \alpha|_{t_0}^{t_1},$$

$$\alpha = \sum_i \frac{\partial L}{\partial v^i} dq^i \in \Omega^1(C).$$

Here EL denotes the term containing the EL equations. By *EL* we will denote the space of solutions to EL.

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Symplectic formulation

$\omega := \mathrm{d}\alpha$ is symplectic iff L is regular. In this case:

- ω is the pullback on $C = TN$ of the canonical symplectic form on T^*N by the Legendre mapping.
- Time evolution is given by a Hamiltonian flow ϕ. In particular,

  $$L := \operatorname{graph} \phi_{t_0}^{t_1} \in \overline{TN} \times TN$$

  is Lagrangian (**canonical relation**).

Remark

L may also be defined directly as $L = \pi(EL)$ with

$$\begin{array}{rccc} \pi : & N^{[t_0,t_1]} & \to & TN \times TN \\ & \{x(t)\} & \mapsto & ((x(t_0), \dot{x}(t_0)), (x(t_1), \dot{x}(t_1))) \end{array}$$

This picture has to be generalized

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Example1: Geodesics

We discuss geodesics on $\mathbb{E}^2$ (Minkowski would be more realistic).

$$L = ||v||,$$

S is defined on $\mathcal{F} := N_0^{[t_0,t_1]} := \{\text{immersed paths}\}$.

- $EL =$ straight lines
- Initial data:
  $\mathcal{F}|_{((t_0))} = \mathbb{R}^2 \times \mathbb{R}^2_* \times \mathbb{R}^\infty = \mathbb{R}^2 \times S^1 \times \mathbb{R}_{>0} \times \mathbb{R}^\infty \ni (\mathbf{q}, \mathbf{v}, \rho, \mathbf{q}_2, \mathbf{q}_3, \dots)$.
- $\alpha = \mathbf{v} \cdot \mathrm{d}\mathbf{q}$
- ω degenerate
- $\tilde{L} := \pi(EL) = \{(\mathbf{q}_1, \mathbf{v}, \rho_1, \dots), (\mathbf{q}_2, \mathbf{v}, \rho_2, \dots)) : \mathbf{q}_1 - \mathbf{q}_2 \,||\, \mathbf{v}\}$
  Not a graph!

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  Not a graph!

Geodesics (continued)

However:

- $\omega|_{\tilde{L}} = 0$, so $\tilde{L}$ is isotropic (actually Lagrangian).
- $\ker\omega = \operatorname{span}\left(\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{q}}, \frac{\partial}{\partial\rho}, \frac{\partial}{\partial\mathbf{q}_2}, \dots\right) =$
  directions parallel to **v**, rescalings of velocity, higher jets;
  so
  $$\varpi\colon \mathcal{F}|_{((t_0))} \to \mathcal{F}^\partial := \mathcal{F}|_{((t_0))}/\ker\omega = TS^1$$
  with canonical symplectic form (identify T and T^* using the metric).
- $L := \varpi(\tilde{L}) = \text{graph Id}$, so a graph and Lagrangian.
- Actually, no time evolution after reduction (an example of topological theory).
- With target $\mathbb{R}^{n+1}$ and Minkowski metric, one gets $\mathcal{F}^\partial = T\mathcal{H}^n$, with $\mathcal{H}^n$ the n-dimensional hyperboloid with induced hyperbolic metric.

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Example 2: 1d Polyakov action

We consider the action

$$S_{[t_0,t_1]} = \int_{t_0}^{t_1} \frac{1}{2} y\dot{x}^2 + \frac{\lambda}{y}$$

where λ is a parameter and the “fields" are $(x, y)\colon [t_0, t_1] \to \mathbb{R} \times \mathbb{R}_{>0}$, so $\mathcal{F} = (\mathbb{R} \times \mathbb{R}_{>0})^{[t_0,t_1]}$.

- For $\lambda > 0$ this is equivalent to the previous example. For $\lambda < 0$ there are no critical points.
- The reduction $\mathcal{F}^\partial$ of the space of boundary jets is $\mathbb{R}^2$ with coordinates (q, p) and $\alpha^\partial = p\,\mathrm{d}q$.
- The map $\pi\colon \mathcal{F} \to \mathcal{F}^\partial \times \mathcal{F}^\partial$ is

$$(x, y) \mapsto (x(t_0), -y(t_0)\,\dot{x}(t_0); x(t_1), -y(t_1)\,\dot{x}(t_1)).$$

- For $\lambda > 0$, we get $L = \{(q_0, 2\lambda, q_1, 2\lambda),\ q_0, q_1 \in \mathbb{R}\}$ which is Lagrangian but not a graph.
- For $\lambda = 0$, we get $L = \{(q, 0, q, 0),\ q \in \mathbb{R}\}$ which is not Lagrangian.

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Example 3: Free 2d particle

- $S_M = \int_M \partial_\mu \phi \, \partial^\mu \phi$ on $\mathbb{R}^M$.
- $EL_M = \{\phi \in \mathbb{R}^M : \Delta\phi = 0\}$.
- Cauchy data (for M a cylinder $S^1 \times I$) $C_{S^1} = (\mathbb{R}^{S^1})^2$: field on S^1 together with its normal derivative.
- If ∂M consistst of n circles $\partial_1 M, \ldots, \partial_n M$:

$$\begin{array}{rccc} \pi : & \mathbb{R}^M & \to & C^n_{S^1} \\ & \phi & \mapsto & ((\phi_{\partial_1 M}, \mathbf{n} \cdot \nabla \phi_{\partial_1 M}), \ldots) \end{array}$$

- $L_M := \pi(EL_M)$ is a graph for M a cylinder, otherwise not a graph.
- However, C_{S^1} is symplectic and L_M is Lagrangian in $C^n_{S^1}$.

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General case

Following ideas by Gawedzki, Schwarz, Fock,. . .

- Let $S_M = \int_M L$ be a class of local actions determined by a Lagrangian L. Here M is a d-manifold.
- S_M is defined on a space of fields F_M (e.g., maps from M to another manifold, connections on M, sections of a fiber bundle,. . . .)

To a $(d-1)$-manifold Σ we associate the space $\tilde{F}_\Sigma$ of germs of fields at $\Sigma \times \{0\}$ on $\Sigma \times [0, \epsilon]$ (“normal derivatives").
The boundary term in the variational calculus defines a one-form $\tilde{\alpha}_\Sigma$ on $\tilde{F}_\Sigma$, for every Σ, with the property

$$\delta S_M = \mathrm{EL}_M + \tilde{\pi}_M^* \tilde{\alpha}_{\partial M},$$

with $\tilde{\pi}_M \colon F_M \to \tilde{F}_{\partial M}$ the natural surjective submersion and EL_M the “EL one-form.”Define $\tilde{\omega}_\Sigma := \mathrm{d}\tilde{\alpha}_\Sigma$.

Assumption

We assume that $\tilde{\omega}_\Sigma$ is presymplectic for every Σ.

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Boundary structure

- Denote by $(F_\Sigma^\partial, \omega_\Sigma^\partial)$ the reduction of $\tilde{F}_\Sigma$ by the kernel of $\tilde{\omega}_\Sigma$.
- For simplicity, we assume that $\tilde{\alpha}_\Sigma$ also descends to a one-form α_Σ on F_Σ^∂.

Then

1. $\omega_\Sigma = \mathrm{d}\alpha_\Sigma$.
2. For every M, we get a projection $\pi_M \colon F_M \to F_{\partial M}^\partial$ and the equation

$$\delta S_M = \mathrm{EL}_M + \pi_M^* \alpha_{\partial M}^\partial$$

Now define $L_M := \pi_M(EL_M)$, which by the previous equation is automatically isotropic.

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Boundary structure (continued)

Assumption

We assume that L_M is Lagrangian for every M.

Remark

This is a requirement for a well-defined theory. It requires, e.g., that YM, CS and *BF* theories should be defined in terms of Lie algebras or the PSM in terms of a Poisson tensor (not just any bivector field).

Definition

For every Σ we define C_Σ as the space of points of F^∂_Σ that can be completed to a pair belonging to $L_{\Sigma\times[0,\epsilon]}$ for some ϵ. In formulae,

$$C_\Sigma = \bigcup_{\epsilon\in(0,+\infty)} L_{\Sigma\times[0,\epsilon]} \circ F^\partial_\Sigma$$

In example 2, $\lambda \geq 0$, $C = \{(q, 2\lambda),\ q \in \mathbb{R}\}$.

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$$C_\Sigma = \bigcup_{\epsilon\in(0,+\infty)} L_{\Sigma\times[0,\epsilon]} \circ F^\partial_\Sigma$$

In example 2, $\lambda \geq 0$, $C = \{(q, 2\lambda),\ q \in \mathbb{R}\}$.

Boundary structure (continued)

Assumption

We assume that L_M is Lagrangian for every M.

Remark

This is a requirement for a well-defined theory. It requires, e.g., that YM, CS and *BF* theories should be defined in terms of Lie algebras or the PSM in terms of a Poisson tensor (not just any bivector field).

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Boundary structure: Reduction

By the assumption, C_Σ is coisotropic. It represents the space of Cauchy data. Its reduction is called the reduced phase space.

Remark

One may consider the symplectic reduction

$$\varpi\colon C_\Sigma \to \underline{C_\Sigma}$$

and also consider the reduced evolution relations

$$\underline{L_M} := \varpi(L_M) \subset \underline{C_{\partial M}}.$$

The reduced phase space is usually very singular. Better to avoid reduction and describe the quotient by some cohomological resolution (BFV).

Boundary structure: composition

Remark (Composition)

If $M = M_1 \cup_\Sigma M_2$, where Σ is (part of) the boundary of M_1 and of M_2,

$$L_M = L_{M_1} \circ L_{M_2} \subset F^{\partial}_{(\partial M_1 \setminus \Sigma) \coprod (\partial M_2 \setminus \Sigma)},$$

where $\circ$ denotes the composition of relations.

Definition

We call $L_{\partial M}$ the **evolution relation**. (More precisely, we split $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$ and regard L_M as a relation in $\overline{F^{\partial}_{(\partial_{\text{in}} M)^{\text{opp}}}} \times F^{\partial}_{\partial_{\text{out}} M}$.)

For a regular theory on a cylinder $M = \Sigma \times I$, L_M is a graph and the composition of cylinders yields the usual composition of maps.

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Boundary structure: composition (continued)

Remark (EL)

By definition the fiber of EL_M over L_M is just one point if M is a short cylinder, but in general it may be much bigger.
So it makes sense to remember it and think of $EL_M \to F^{\partial}_{\partial_M}$ as a correspondence, the **evolution correspondence**.

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Gauge theories

- If $C_\Sigma \subset F^\partial_\Sigma$ (proper subset!), we say that S defines a gauge theory.
- Notice that L_M is not a graph, even if M is a cylinder. In particular,

  $$R_\Sigma := \text{``}\lim_{\epsilon\to 0}\text{''}\, L_{\Sigma\times[0,\epsilon]} \subset \overline{C_\Sigma} \times C_\Sigma$$

  is not a graph.
  It is an equivalence relation (gauge transformation) in C_Σ and

  $$\underline{C_\Sigma} = C_\Sigma / R_\Sigma .$$

A topological field theory is a Lagrangian field theory that is invariant under diffeomorphisms.
So, in particular, it is a gauge theory and moreover

$$\underline{L_{\Sigma\times I}} = \text{graph}(\text{Id}_{\underline{C_\Sigma}})$$

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One usually also requires all $\underline{C_\Sigma}$s to be finite dimensional (sometimes even compact).

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Axiomatics

We may then think of a classical Lagrangian field theory in d dimensions as the following data:

- A space of field F_M for every d-manifold M
- A symplectic space F^∂_Σ for every $(d-1)$-manifold Σ
- A Lagrangian correspondence $\pi\colon EL_M \to F^\partial_{\partial M}$ for every M.
- $(F_\bullet, C_\bullet)$ should be thought as a functor.

It may be good to assume that the fibers of the correspondence are finite dimensional.

Remark

In the reduced picture (in case of trivial fibers), the target “category" is that of (singular) symplectic manifolds and canonical relations. Notice that the reduced evolution relation for a (short) cylinder is a graph, actually a flow. In particular,

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Quantization of regular Lagrangian field theories

- In a regular theory, $\underline{C_\Sigma} = C_\Sigma = F^\partial_\Sigma$ is symplectic; geometric quantization: vector space H_Σ.
- For simplicity, assume that the symplectic manifold $C_{\partial M}$ is endowed with a Lagrangian foliation along which $\alpha_{\partial M}$ vanishes and with a smooth leaf space $B_{\partial M}$. (One may change $\alpha_{\partial M}$ to this goal.) Then $H_{\partial M}$ is a space of functions on $B_{\partial M}$.Denote by $p_{\partial M}$ the projection $C_{\partial M} \to B_{\partial M}$.
- The canonical relation $L_M \subset C_{\partial M}$ is quantized to a state $\psi_M \in H_{\partial M}$. Asymptotically,

$$\psi_M(\varphi) = \int_{\Phi \in \pi_M^{-1}(p_{\partial M}^{-1}(\varphi))} e^{\frac{i}{\hbar} S_M(\Phi)}\,[D\Phi], \qquad \varphi \in B_{\partial M}$$

- If $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$, then $\psi_M \in H^*_{\partial_{\text{in}} M} \otimes H_{\partial_{\text{out}} M}$.
  Hence, operator $H_{\partial_{\text{in}} M} \to H_{\partial_{\text{out}} M}$.
  Composition of relations goes to composition of operators.
- Cfr. Segal's axiomatization of CFT and Atiyah's axiomatization of TFT.

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The BV construction

- In a gauge theory, this does not work since EL_M is still infinite dimensional. One introduces “symmetries” s.t. the quotient $\underline{EL_M}$ is finite dimensional (often discrete). But: Too complicated (and often singular) to perform the Gaussian perturbative expansion.
- If M has no boundary, the Batalin–Vilkovisky (BV) construction yields a BV manifold $(\mathcal{F}_M, \omega_M, S_M)$, where
  1. $\mathcal{F}_M$ is a supermanifold with additional $\mathbb{Z}$-grading (containing the original F_M as its degree zero component).
  2. ω_M is a symplectic form of degree -1 on $\mathcal{F}_M$.
  3. S_M is a function of degree zero on $\mathcal{F}_M$ which extends the classical action and satisfies the CME

$$\{S_M, S_M\} = 0.$$

One defines Q_M as the Hamiltonian vector field of S_M

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M$$

Q_M has degree one and $[Q_M, Q_M] = 0$ (cohomological vector field). Its zero locus $\mathcal{EL}_M$ is the same as the critical set of S_M.

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The case with boundary

- The equation
$$\iota_{Q_M}\omega_M = \mathrm{d}S_M$$
no longer holds if M has boundary. We have to deal with the boundary terms in computing $\mathrm{d}S_M$ as in the first part of this talk.
- Define the space $\tilde{\mathcal{F}}_\Sigma$ of preboundary fields on a $(d-1)$-manifold Σ as the germs at $\Sigma \times \{0\}$ of $\mathcal{F}_{\Sigma\times[0,\epsilon]}$. Integration by parts in the computation of $\mathrm{d}S_{\Sigma\times[0,\epsilon]}$ yields a one-form $\tilde{\alpha}_\Sigma$ of degree zero on $\tilde{\mathcal{F}}_\Sigma$. We denote by $\tilde{\omega}_\Sigma$ its differential.

Assumption

We assume that $\tilde{\omega}_\Sigma$ is presymplectic.

- Denote by $(\mathcal{F}^\partial_\Sigma, \omega^\partial_\Sigma)$ the reduction of $(\tilde{\mathcal{F}}_\Sigma, \tilde{\omega}_\Sigma)$.
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The case with boundary (continued)

Let $\pi_M \colon \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$ be the induced surjective submersion.
One can then prove that

1. Q_M descends to a cohomological vector field $Q^\partial_{\partial M}$ which is Hamiltonian w.r.t. $\omega^\partial_{\partial M}$.

Remark

One then says that the triple $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M}, Q^\partial_{\partial M})$ is a BFV manifold. Notice that the degree of $\omega^\partial_{\partial M}$ is now zero. The zero locus of $Q^\partial_{\partial M}$ is coisotropic. Its degree zero component $C_{\partial M}$ is also coisotropic. If its reduction is smooth, its Poisson algebra of functions is the same as the cohomology of $Q^\partial_{\partial M}$ in degree zero. The BFV construction has to be thought of as a resolution of this quotient.

2. We have the fundamental equation of the BV theory for manifolds with boundary [C, Mnëv, Reshetikhin]:

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Example: Electromagnetism

- Maxwell's equations: $\mathrm{d}^*\mathrm{d}A = 0$, A connection 1-form.
- First-order formalism: $S_M^{\mathrm{cl}} = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B$
  B a $(d-2)$-form. Then $EL = \{*B = \mathrm{d}A,\ \mathrm{d}B = 0\}$.
- BV: $S_M = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B + A^+\,\mathrm{d}c$
  A^+: $(d-1)$-form, ghost number -1; c: 0-form, ghost number 1.
  $\omega_M = \int_M \delta A\,\delta A^+ + \delta B\,\delta B^+ + \delta c\,\delta c^+$,
  B^+ and c^+ do not show up in the action.
  $QA = \mathrm{d}c,\ QA^+ = \mathrm{d}B,\ QB^+ = *B + \mathrm{d}A,\ Qc^+ = \mathrm{d}A^+$.
- Boundary fields: A, B, A^+, c,
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  Interpretation:
  $A =$ vector potential, up to gauge transformations $A \mapsto A + \mathrm{d}c$
  $B =$ electric field constrained by Gauss law $\mathrm{d}B = 0$.

Example: Electromagnetism

- Maxwell's equations: $\mathrm{d}^*\mathrm{d}A = 0$, A connection 1-form.
- First-order formalism: $S_M^{\text{cl}} = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B$
  B a $(d-2)$-form. Then $EL = \{*B = \mathrm{d}A,\ \mathrm{d}B = 0\}$.
- BV: $S_M = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B + A^+\,\mathrm{d}c$
  A^+: $(d-1)$-form, ghost number -1; c: 0-form, ghost number 1.
  $\omega_M = \int_M \delta A\,\delta A^+ + \delta B\,\delta B^+ + \delta c\,\delta c^+$,
  B^+ and c^+ do not show up in the action.
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Properties

The fundamental equation

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M + \pi_M^*\alpha^\partial_{\partial M} \tag{1}$$

has several consequences:

1. $L_{Q_M}\omega_M = \pi_M^*\omega^\partial_{\partial M}$ (Q_M not symplectic).
2. $Q_M(S_M) = \pi_M^*(2S^\partial_{\partial M} - \iota_{Q^\partial_{\partial M}}\alpha^\partial_{\partial M})$ (modified CME).
3. $\mathcal{E}\mathcal{L}_M :=$ {zeros of Q_M} coisotropic,

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   Then $\mathcal{E}_\ell$ presymplectic and we have a fibration $\underline{\mathcal{E}\mathcal{L}_M} \to \underline{\mathcal{L}_M}$ with finite dimensional odd symplectic fiber $\underline{\mathcal{E}_\ell}$ over $\underline{\ell}$.
   BV canonical correspondence

Example EM:
$\underline{\mathcal{E}_\ell} = H^1(M,\partial M) \oplus H^{n-1}(M)[-1] \oplus H^0(M,\partial M)[1] \oplus H^n(M)[-2]$

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Boundaries of boundaries

- On every boundary component Σ, we now have a BFV manifold $(\mathcal{F}^\partial_\Sigma, \omega^\partial_\Sigma, Q^\partial_\Sigma)$. Assume it is given by local data. Let S^∂_Σ be the Hamiltonian function of Q^∂_Σ: $\iota_{Q^\partial_\Sigma}\omega^\partial_\Sigma = \mathrm{d}S^\partial_\Sigma$.
- If Σ has a boundary γ, we may repeat the previous construction verbatim. We get
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  3. and so on.

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It makes sense however to stop if the fibers of the correpondences become infinite dimensional. In TFTs and in 2d YM one can go down up to dimension zero (fully extended field theories).

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Quantization

1. Fix a polarization on $\mathcal{F}^{\partial}_{\partial M}$ such the quantization $\Omega_{\partial M}$ of $S^{\partial}_{\partial M}$ squares to zero.
2. For simplicity, assume we have a transversal $\mathcal{L}'$ to the polarization. So $\mathcal{H}_{\partial M} =$ functions on $\mathcal{L}'$.
3. Define
$$\psi_M = \int e^{\frac{i}{\hbar} S_M} \in \mathcal{H}_{\partial M}$$
where the integral is over a Lagrangian submanifold of the fiber over a boundary field in $\mathcal{L}'$.
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Perturbative quantization

Usually, the only way of computing the functional integral is to perturb around a Gaussian theory.
Let S^0 be the Gaussian theory and denote by $\mathcal{Z}^0_M$ the space of functions on the fiber of $\underline{\mathcal{EL}}^0_M$ (“vacua"). Then

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$$\psi_M = \int e^{\frac{i}{\hbar} S_M} \in \mathcal{H}_{\partial M} \otimes \mathcal{Z}^0_M$$
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Axiomatics

- To each $(d-1)$-manifold Σ we associate a complex $(\mathcal{H}_\Sigma, \Omega_\Sigma)$.
- To each d-manifold we as associate a f.d. BV manifold $\underline{\mathcal{EL}}_M$ ("moduli space of vacua"), the BV algebra $\mathcal{Z}_M$ of functions on $\underline{\mathcal{EL}}_M$ (endowed with a BV operator Δ), and an element ψ_M of $\mathcal{H}_{\partial M} \otimes \mathcal{Z}_M$ satisfying the modified QME.
- Plus functorial properties.

Eventually, we may integrate over a Lagrangian submanifold of $\underline{\mathcal{EL}}_M$ and go to the Ω_Σ-cohomology getting just a state in the physical space.

Remark

The full power of this approach is that we may cut the original manifold M into simple, or tiny, pieces; do the perturbative quantization there; and eventually glue and reduce.
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$$S = \int_M \left\langle B, \, \mathrm{d}A + \tfrac{1}{2}[A, A] \right\rangle, \; A \in \Omega(M, \mathfrak{g}), \; B \in \Omega(M, \mathfrak{g}^*)$$

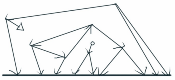
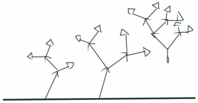


Figure: $\frac{\delta}{\delta B}$-foliation



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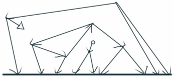

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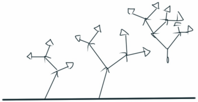

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