Universität Hamburg
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Complex functions for students of engineering sciences

Auditorium exercise 3:

The complex logarithm, the Joukowski function, Möbius transformations

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1. The complex logarithm

The complex logarithm is a **set valued** function, of which we mostly look at the **principal value**.

$$e^{\log(|z|) + i(\arg(z) + 2k\pi)}$$
$$= e^{\log(|z|)} \cdot e^{i(\arg(z)+2k\pi)}$$
$$= |z| \, e^{i(\arg(z)+2k\pi)} = z$$

For $z \in \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$, $\quad z = |z| \cdot \mathrm{e}^{\mathrm{i}\arg(z)}$:

$$\left\{ \ \mathrm{Log}(z) \ \right\} = \left\{ \ \log(|z|) + \mathrm{i}\,(\arg(z) + 2k\pi) \mid k \in \mathbb{Z} \ \right\}.$$

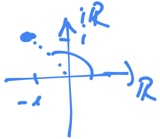
For $\arg(z) \in (-\pi, \pi)$ we call $\log(|z|) + \mathrm{i}\arg(z)$ the **principal value** of the complex logarithm.

The function

$$\ln(z) \quad \mathrm{Log}(z) = \log(|z|) + \mathrm{i}\arg(z), \qquad \text{mit} \quad \arg(z) \in (-\pi, \pi),$$

is called the **principal branch** of the complex logarithm.

Often this is denoted by $\ln(z)$.



Principal value:

Example: $z \in \mathbb{R},\ z > 0$: $\quad \mathrm{Log}(z) = \log(z)$ (real logarithm).

Example: $z_1 = \sqrt{2}(-1 + \mathrm{i}), \qquad z_2 = 3\mathrm{i}, \qquad z_3 = -4\mathrm{i}.$

$$z_1 = \sqrt{2}\cdot\sqrt{2}\, e^{i\frac{3\pi}{4}} = 2e^{i\frac{3\pi}{4}} \Rightarrow \{\mathrm{Log}(z_1)\} = \Bigl\{ \underbrace{\log(2) + i\,\bigl(\tfrac{3\pi}{4}}_{\text{principal value}} + 2k\pi\bigr) \;\Big|\; k\in\mathbb{Z} \Bigr\}$$

From now on: Only principal values!

$$z_2 = 3e^{i\frac{\pi}{2}} \Rightarrow \mathrm{Log}(z_2) = \log(3) + i\,\frac{\pi}{2}$$

$$z_3 = 4e^{-i\frac{\pi}{2}} \Rightarrow \mathrm{Log}(z_3) = \log(4) + i\left(-\frac{\pi}{2}\right)$$

[$= 4e^{i\frac{3\pi}{2}}$, but then the angle is not in $(-\pi, \pi)$]

Example: $z_1 = 2\exp\left(i\frac{3\pi}{4}\right), \qquad z_2 = 3i, \qquad z_3 = -4i.$

not in $(-\pi, \pi)$

$$z_1 \cdot z_2 = 6\exp\left(i\frac{5\pi}{4}\right) = 6\exp\left(i\left(-\frac{3\pi}{4}\right)\right)$$

$$\Rightarrow \operatorname{Log}(z_1 \cdot z_2) = \log(6) + i\frac{5\pi}{4}$$

$$\frac{z_1}{z_2} = \frac{2}{3}\exp\left(i\frac{\pi}{4}\right) \Rightarrow \operatorname{Log}\left(\frac{z_1}{z_2}\right) = \log\left(\frac{2}{3}\right) + i\frac{\pi}{4}$$

$$z_1 \cdot z_3 = 8\exp\left(i\frac{\pi}{4}\right) \Rightarrow \log(z_1 \cdot z_3) = \log(8) + i\frac{\pi}{4}$$

$$\frac{z_1}{z_3} = \frac{1}{2}\exp\left(i\left(-\frac{3\pi}{4}\right)\right) \Rightarrow \log\left(\frac{z_1}{z_3}\right) = \log\left(\frac{1}{2}\right) + i\left(-\frac{3\pi}{4}\right)$$

Be careful with the rules for the real logarithm!

They don't generalize directly to the complex case!

For the **real logarithm** it holds with $a, b > 0$:

$$\log(a \cdot b) = \log(a) + \log(b), \qquad \log\left(\frac{a}{b}\right) = \log(a) - \log(b).$$

For the **principal branch of the complex logarithm** this is, in general, not true!

In general $\operatorname{Log}(z_1 \cdot z_2) \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$

However, it still holds that

$$\exp\left(\operatorname{Log}(z_1) + \operatorname{Log}(z_2)\right) = \exp\left(\operatorname{Log}(z_1)\right) \cdot \exp\left(\operatorname{Log}(z_2)\right) = z_1 \cdot z_2.$$

$$\operatorname{Log}(z_1) + \operatorname{Log}(z_2) \in \{\operatorname{Log}(z_1 \cdot z_2)\}$$

We've computed a minute ago:

$$\mathrm{Log}(z_1) = \log(2)+\mathrm{i}\frac{3\pi}{4}, \qquad \mathrm{Log}(z_2) = \log(3)+\mathrm{i}\frac{\pi}{2}, \qquad \mathrm{Log}(z_3) = \log(4)+\mathrm{i}\left(-\frac{\pi}{2}\right)$$

With that:

$\log(2) + \log(3)$
$= \log(2 \cdot 3) = \log(6)$

This always works out.

This can be different.

$$\mathrm{Log}(z_1 \cdot z_2) = \log(6) - \mathrm{i}\frac{3\pi}{4} \neq \mathrm{Log}(z_1) + \mathrm{Log}(z_2) = \log(6) + i\,\frac{5\pi}{4}$$

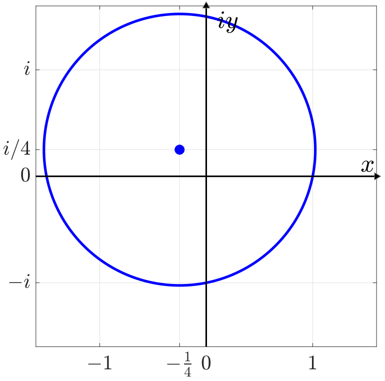
$$\mathrm{Log}(z_1 \cdot z_3) = \log(8) + \mathrm{i}\frac{\pi}{4} = \mathrm{Log}(z_1) + \mathrm{Log}(z_3) = \log(8) + i\,\frac{\pi}{4}$$

$$\mathrm{Log}(z_1/z_2) = \log\left(\frac{2}{3}\right) + \mathrm{i}\frac{\pi}{4} = \mathrm{Log}(z_1) - \mathrm{Log}(z_2) = \log\left(\frac{2}{3}\right) + i\,\frac{\pi}{4}$$

$$\mathrm{Log}(z_1/z_3) = \log\left(\frac{1}{2}\right) - \mathrm{i}\frac{3\pi}{4} \neq \mathrm{Log}(z_1) - \mathrm{Log}(z_3) = \log\left(\frac{1}{2}\right) + i\,\frac{5\pi}{4}$$

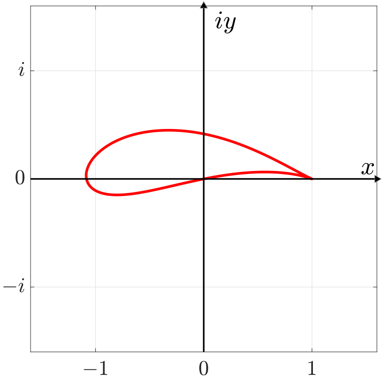
2. The Joukowski function

The Joukowski function has applications in aerodynamics.



$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\longrightarrow$$



$z_0 = -\frac{1}{4}(-1+\mathrm{i}), \quad r_0 = \frac{\sqrt{26}}{4}$

1 on the circle,
-1 in the interior

Looks a bit like a wing
of an airplane

Real and imaginary part of the Joukowski funktion

$$\frac{1}{re^{i\varphi}} = \frac{1}{r} e^{-i\varphi}$$
$$\cos(-\varphi) = \cos(\varphi)$$
$$\sin(-\varphi) = -\sin(\varphi)$$

$$w \;=\; f(z) \;=\; \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad z \neq 0.$$

with

$$z \;=\; re^{i\varphi}, \qquad w \;=\; u + \mathrm{i}v$$

we get

$$u \;=\; \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\varphi), \qquad v \;=\; \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\varphi).$$

And for $z = x + \mathrm{i}y$: $x = r\cos(\varphi)$, $y = r\sin(\varphi)$, $\frac{1}{r} = \frac{r}{r^2}$

$$u \;=\; \frac{1}{2}\left(x + \frac{x}{x^2+y^2}\right), \qquad v \;=\; \frac{1}{2}\left(y - \frac{y}{\underbrace{x^2+y^2}_{r^2}}\right).$$

Images of circles

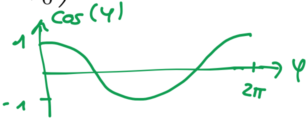
$E = \{z \in \mathbb{C} \mid |z| = 1\}$

$\Rightarrow f(E) = [-1, 1]$

For fixed $\quad r = r_0, \quad \varphi \in [0, 2\pi):$

$$u = \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right)\cos(\varphi), \qquad v = \frac{1}{2}\left(r_0 - \frac{1}{r_0}\right)\sin(\varphi).$$

For $\quad r_0 = 1: \qquad u = \cos(\varphi), \qquad v = 0.$



we run through $[-1, 1]$ twice!

For $\quad r_0 \neq 1:$

$$\frac{u^2}{\frac{1}{4}(r_0 + \frac{1}{r_0})^2} + \frac{v^2}{\frac{1}{4}(r_0 - \frac{1}{r_0})^2} = 1.$$

$$= \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

ellipse

Images of beams

For fixed $\varphi_0 \in [0, 2\pi)$, $r > 0$:

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\varphi_0), \qquad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\varphi_0).$$

Analyse the functions

$$\lim_{r \to 0} g(r) = \infty = \lim_{r \to \infty} g(r)$$

$$g'(r) = \frac{1}{2} - \frac{1}{2r^2} \stackrel{!}{=} 0 \Rightarrow r = 1$$

$$g''(r) = \frac{1}{r^3} > 0 \text{ for } r > 0 \Rightarrow \text{Minimum at } g(1) = 1.$$

$$g(\mathbb{R}_+) = [1, \infty)$$

$\varphi_0 = \frac{\pi}{2}$:

$$\lim_{r \to \infty} h(r) = \infty$$

$$g(r) := \frac{1}{2}\left(r + \frac{1}{r}\right):$$

$$\lim_{r \to 0} h(r) = -\infty$$

$$\Rightarrow f(i\mathbb{R}) = i\mathbb{R}$$

$$h(r) := \frac{1}{2}\left(r - \frac{1}{r}\right):$$

For $\varphi_0 = 0$: $f(\mathbb{R}_+) = [1, \infty)$

The inverse Joukowski function

The Joukowski function is not invertible on the unit circle, because

$$f(\mathrm{e}^{i\varphi}) \;=\; \frac{1}{2}\left(\mathrm{e}^{i\varphi} + \mathrm{e}^{-\mathrm{i}\varphi}\right) \;=\; f(e^{-\mathrm{i}\varphi}).$$

For $|z| > 1$ we can compute the inverse directly:

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \Rightarrow \qquad z^2 - 2wz + 1 = 0 \qquad \Rightarrow \qquad z = w \pm \sqrt{w^2 - 1},$$

where we choose the branch of the square root, such that $|z| > 1$.

3. Möbius transformations

The extended complex plane and generalized circles

We add the **„infinite point“** ∞ to the complex plane, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

A **generalized circle** in $\mathbb{C}^*$ is either

- a (true) circle in $\mathbb{C}$,
- a straight line through ∞.

All lines go through this point.

Möbius transformations

A **Möbius transformation** it a function

$$T : \mathbb{C}^* \to \mathbb{C}^*, \qquad T(z) = \frac{az+b}{cz+d}$$

with

$$a,\ b,\ c,\ d \in \mathbb{C}, \qquad \text{und} \qquad \underline{ad - bc} \neq 0.$$

↳ Nominator and denominator have different zeros.

If $z_1,\ z_2,\ z_3 \in \mathbb{C}^*$ and $w_1,\ w_2,\ w_3 \in \mathbb{C}^*$, are, respectively, pairwise distinct, then T is uniquely determined by the **interpolation condition**

$$T(z_j) = w_j, \qquad j = 1, 2, 3.$$

This is easy if we know a **zero** and a **pole**.

For given z_1 and z_2 with $T(z_1) = 0$, $T(z_2) = \infty$, the Möbius transformation T has the form

$$T(z) = \alpha \cdot \frac{z - z_1}{z - z_2}, \qquad \alpha \in \mathbb{C}.$$

Example:

$$T(\mathrm{i}) = 0, \qquad T(-2\mathrm{i}) = \infty, \qquad T(2) = 3 + \mathrm{i} \qquad \Rightarrow \qquad T(z) = \alpha \cdot \frac{z - \mathrm{i}}{z + 2\mathrm{i}}.$$

Finde α :

$$T(2) = \alpha \cdot \frac{2 - \mathrm{i}}{2 + 2\mathrm{i}} = \frac{\alpha}{4}(2 - \mathrm{i})(2 - 2\mathrm{i})$$

$$= \frac{\alpha}{4}(1 - 3\mathrm{i}) \overset{!}{=} 3 + \mathrm{i} \qquad \Rightarrow \qquad \alpha = 4\mathrm{i}$$

$$\Rightarrow \qquad T(z) = \frac{4\mathrm{i} \cdot z + 4}{z + 2\mathrm{i}}.$$

We can find the interpolating Möbius transformation by the **three point formula**.

For pairwise distinct $z_1,\ z_2,\ z_3 \in \mathbb{C}^*$ and $w_1,\ w_2,\ w_3 \in \mathbb{C}^*$, solve

$$\frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = \frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2}$$

for w and let $T(z) = w$.

Then T satisfies

$$T(z_j) = w_j, \qquad j = 1, 2, 3.$$

We can find the interpolating Möbius transformation by the **three point formula**.

Example:

$$z_1 = 0, \qquad z_2 = -2, \qquad z_3 = -1 + \mathrm{i},$$
$$w_1 = 0, \qquad w_2 = 4, \qquad w_3 = 2 + 2\mathrm{i}.$$

Use the tree point formula:

$$\frac{w - 0}{w - 4} : \frac{(2+2\mathrm{i}) - 0}{(2+2\mathrm{i}) - 4} = \frac{z - 0}{z - (-2)} : \frac{(-1+\mathrm{i}) - 0}{(-1+\mathrm{i}) - (-2)}$$

$$\Leftrightarrow \quad \frac{w}{w - 4} : \frac{\mathrm{i}+1}{\mathrm{i}-1} = \frac{z}{z+2} : \frac{\mathrm{i}-1}{\mathrm{i}+1}$$

$$\Leftrightarrow \quad w(z+2) = z(w-4)\underbrace{\frac{(\mathrm{i}+1)^2}{(\mathrm{i}-1)^2}}_{= -1} \quad \Leftrightarrow \quad wz + 2w = 4z - wz$$

$$\Leftrightarrow \quad w = \frac{2z}{z+1} := T(z).$$

Möbius transformations are circle preserving.

Let:

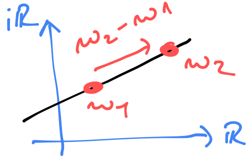
- $T(z) = \dfrac{az+b}{cz+d}, \qquad ad - bc \neq 0,$ a Möbius transformation;
- $K \subset \mathbb{C}^*$ a generalized circle.

$$T\left(-\frac{d}{c}\right) = \frac{a\left(-\frac{d}{c}\right)+b}{c\left(-\frac{d}{c}\right)+d} = \frac{k}{0} = \infty$$

Then the circle preservation holds:

- $-\dfrac{d}{c} \in K \quad \Rightarrow \quad T(K)$ is a straight line. ($T(K)$ is a generalized circle!)
- $-\dfrac{d}{c} \notin K \quad \Rightarrow \quad T(K)$ is a circle.

How do we find images of generalized circles?



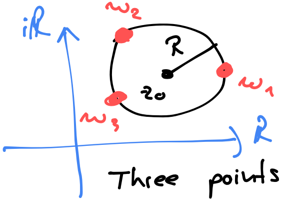
Two points determine a line.

If $\quad -\dfrac{d}{c} \in K \quad$ (image is a straight line):

- Chose $\quad z_1,\ z_2 \in K$, compute $\quad w_1 = T(z_1),\ w_2 = T(z_2)$;
- Image line: $\quad T(K) = \{w \in \mathbb{C} \mid w = w_1 + \alpha(w_2 - w_1),\ \alpha \in \mathbb{R}\} \cup \{\infty\}$.

Often it is easier: Use symmetry!

How do we find images of generalized circles?



Three points determine a circle

If $\quad -\dfrac{d}{c} \notin K \quad$ (image is a true circle):

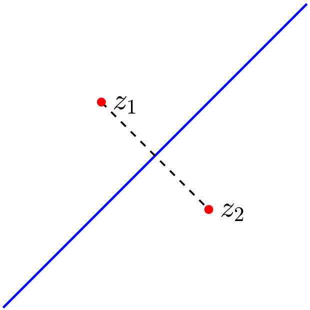
- Choose $\quad z_1,\ z_2,\ z_3 \in K$, compute $w_1 = T(z_1),\ w_2 = T(z_2),\ w_3 = T(z_3)$;
- Solve

$$(1)\quad |w_1 - w_0|^2 = R^2, \qquad (2)\quad |w_2 - w_0|^2 = R^2, \qquad (3)\quad |w_3 - w_0|^2 = R^2$$

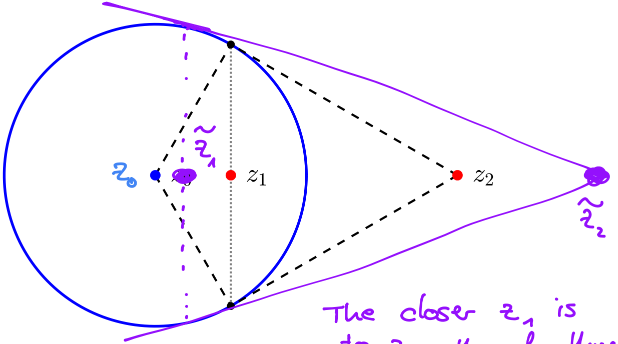
  for $\quad w_0 = u_0 + iv_0 \quad$ and R.
- Image circle: $\quad T(K) = \{w \in \mathbb{C} \mid |w - w_0| = R\}$.

Often it is easier: Use symmetry!

Symmetries w.r.t. generalized circles



Symmetry w.r.t. a line:
Reflection



The closer z_1 is to z_0, the further outside is z_2.

Symmetry w.r.t. a circle
with center z_0 and radius R:

$$(z_1 - z_0)(\overline{z_2} - \overline{z_0}) = R^2.$$

We agree: z_0 is symmetric to ∞.

Möbius transformations **preserve symmetry** w.r.t. generalized circles.

Let:

- $T : \mathbb{C}^* \to \mathbb{C}^*$ a Möbius transformation;
- $K \subset \mathbb{C}^*$ a generalized circle;
- $z_1,\ z_2 \in \mathbb{C}^*$.

Then the **symmetry preservation** holds:

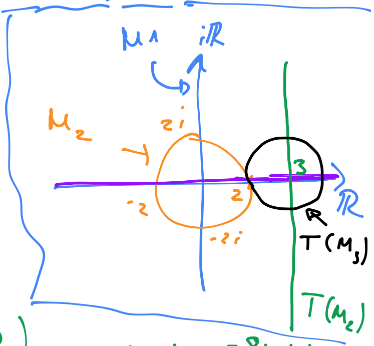
$$z_1,\ z_2 \quad \text{symmetric w.r.t.} \quad K$$

$$\Rightarrow \qquad T(z_1),\ T(z_2) \quad \text{symmetric w.r.t.} \quad T(K)$$

Example: Möbius transformation $T(z) = \dfrac{4z - 4i}{z - 2i}$ $\qquad -\frac{d}{c} = 2i$

M_1: imaginary axis : $2i \notin M_1 \Rightarrow T(M_1)$ is a straight line.

$$T(iy) = \frac{4iy - 4i}{iy - 2i} = \frac{4y - 4}{y - 2}$$: all coeff. real $\Rightarrow$ image is the whole real axis



M_2: circle $|z| = 2$

$2i \in M_2 \Rightarrow M_2$ is a straight line.

M_2 symm. w.r.t $M_1 \Rightarrow T(M_2)$ symm. w.r.t. $T(M_1) = \mathbb{R}$

$\Rightarrow T(M_2)$ orthogonal to $\mathbb{R}$ (parallel to $i\mathbb{R}$). $\quad T(-2i) = \dfrac{-8i - 4i}{-2i - 2i} = 3$

$\Rightarrow T(M_2) = \{ z \in \mathbb{C} \mid \operatorname{Re}(z) = 3 \}$

M_3: real axis

$2i \notin M_3 \Rightarrow T(M_3)$ true circle

M_3 symm. to M_1 and $M_2 \Rightarrow T(M_3)$ symm. to $T(M_1)$ and $T(M_2)$

$\Rightarrow$ center at 3. Radius : $R = |T(0) - 3| = \left| \frac{-4i}{-2i} - 3 \right| = 1.$

Example: Möbius transformation $T(z) = \dfrac{3z - \mathrm{i}}{z + 5\mathrm{i}}$ $-\frac{d}{c} = -5i$

M: circle $|z - 4i| = 3$, $-5i \notin M \Rightarrow T(M)$ true circle.

Let $w_* = T(z_*)$ be the center. Then

$w_* = T(z_*)$ symm. to $\infty = T(-5i)$ w.r.t. $T(M)$

$\Rightarrow$ z_* symm. to $-5i$ w.r.t. M

$\Rightarrow$ $9 = (z_* - 4i)(5i + 4i)$ $= (z_* - 4i) \cdot 9i \Rightarrow z_* - 4i = -i$

$\left[R^2 = (z_* - z_0)(\overline{-5i} - \overline{z_0})\right]$ $\Rightarrow z_* = 3i$

$\Rightarrow w_* = T(3i) = \dfrac{9i - i}{3i + 5i} = \dfrac{8i}{8i} = 1$ is the center of $T(M)$.

Radius: $i \in M$, $R = |T(i) - 1| = \left|\dfrac{3i - i}{i + 5i} - 1\right| = \left|\dfrac{2}{6} - 1\right| = \dfrac{2}{3}$

$T(M) = \{z \in \mathbb{C} \mid |z - 1| = \frac{2}{3}\}$.