Universität Hamburg
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# Complex functions
# for students of engineering sciences

## Auditorium exercise 2:

## Elementary complex functions,
## natural powers and roots

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# 1. Elementary complex functions

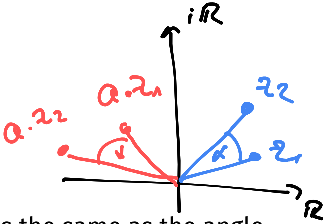
# Linear and affine functions

$$f : \mathbb{C} \to \mathbb{C}, \qquad f(z) \;=\; a \cdot z \;+\; b, \qquad \text{with given} \quad a \in \mathbb{C} \setminus \{0\}, \quad b \in \mathbb{C}$$

We already know

- Addition $(+b)$ means shifting in the direction of $b$.
- Multiplication $(a \cdot z)$ means rotating by $\arg(a)$ and stretching by a factor of $|a|$:
  With $\quad a = |a|\mathrm{e}^{\mathrm{i}\arg(a)}, \quad z = r\mathrm{e}^{\mathrm{i}\varphi}$:

$$a \cdot z \;=\; |a| r \cdot \mathrm{e}^{\mathrm{i}(\varphi + \arg(a))}$$




$\longrightarrow$ **All $z \in \mathbb{C}$ are rotated by the same angle and stretched by the same factor!**

In particular: For a fixed $a \in \mathbb{C}$ the angle between $z_1, z_2 \in \mathbb{C}$ is the same as the angle between $a \cdot z_1$ and $a \cdot z_2$!

$\mathbb{C}$-linear maps preserve angles.

Linear mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ can do a lot more!

$A x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

**Example:** $f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x) = Ax, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

Then $\|x_1\|_2 = \|x_2\|_2 = 1$, but $\|Ax_1\|_2 = 1$, $\|Ax_2\|_2 = \sqrt{2}$, d.h. so different directions get stretched by a different factor!

This map does an **anisotropic** stretching.

**Example:** $f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x) = Ax, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

The angle between $x_1$ and $x_2$ is $\pi/2$, but the angle between $Ax_1 = (1,0)^\top$ and $Ax_2 = (1,1)^\top$ is $\pi/4$.

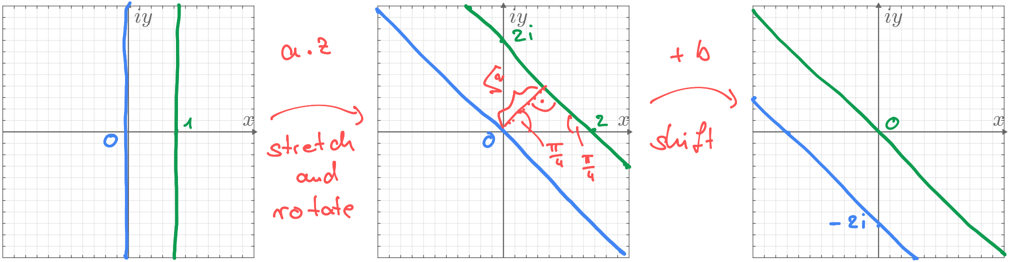
This mapping is **not angle preserving**.

Linearity in $\mathbb{C}$ has stronger implications than linearity in $\mathbb{R}^2$. This will become important when we discuss differentiability in $\mathbb{C}$.

# Images under linear functions

**Example:** $f(z) = a \cdot z + b, \qquad a = 1 + \mathrm{i} = \sqrt{2}e^{i\frac{\pi}{4}}, \qquad b = -2\mathrm{i}$

$$M = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$$



The function $f(z) = z^{-1}$ $= \frac{1}{z}$

$r = 1 \Rightarrow \frac{1}{r} = 1$

For $z \neq 0,\ z = re^{i\varphi}$: $\quad z^{-1} = \frac{1}{z} = \frac{1}{r}e^{-i\varphi}$.

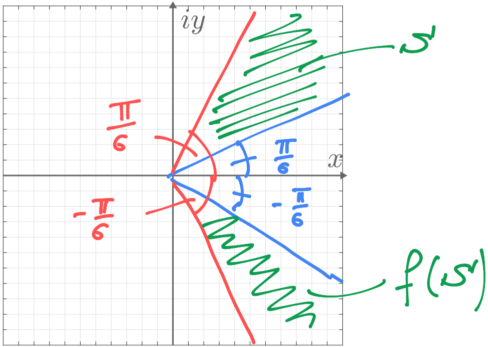
$r = \frac{1}{2} \Rightarrow \frac{1}{r} = 2$

**Example:**

$-\varphi \in [-\frac{\pi}{3}, -\frac{\pi}{6}]$

$$S = \left\{ re^{i\varphi} \in \mathbb{C} \mid r > 0, \quad \varphi \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \right\}$$

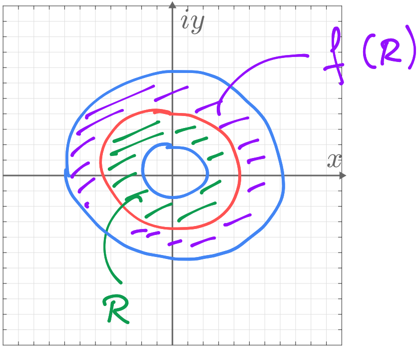
$\{r > 0\} = (0, \infty) \Rightarrow \{\frac{1}{r}\} = (0, \infty)$




**Example:**

$-\varphi \in [-\pi, \pi)$

$$R = \left\{ re^{i\varphi} \in \mathbb{C} \mid r \in [0.5, 1], \quad \varphi \in (-\pi, \pi] \right\}$$



# Interlude: Representing circles

Recall: $\mathcal{K} = \{z \in \mathbb{C} \mid |z - z_0| = R\}$

$= \{z \in \mathbb{C} \mid z = z_0 + Re^{i\varphi},\ \varphi \in [0, 2\pi)\}$

Circle with center $z_0 \in \mathbb{C}$, radius $R > 0$:

$$|z - z_0| = R \quad \Leftrightarrow \quad |z - z_0|^2 = R^2$$

With that:

$$\begin{aligned} R^2 &= |z - z_0|^2 = (z - z_0)\overline{(z - z_0)} \\ &= (z - z_0)(\overline{z} - \overline{z_0}) \\ &= z\overline{z} - z\overline{z_0} - z_0\overline{z} + z_0\overline{z_0} \end{aligned}$$

Back to $f(z) = z^{-1}$.

**Example:** Circle with center $z_0 = 2$, radius $2$, without zero:

$$D = \{z \in \mathbb{C} \setminus \{0\} \mid |z-2| = 2\}.$$

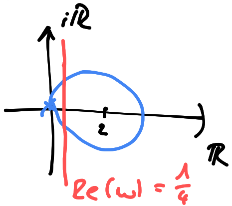
What is $f(D)$ for $f(z) = z^{-1}$ ?

With the representation above: $|z-2|^2 = z\overline{z} - 2z - 2\overline{z} \cancel{+4} = \overset{0}{\cancel{4}}.$

Consider $w = \frac{1}{z} \Leftrightarrow z = \frac{1}{w}$ : $\quad \cdot\, w\overline{w} \neq 0$

$$\frac{1}{w}\frac{1}{\overline{w}} - 2\frac{1}{w} - 2\frac{1}{\overline{w}} = 0 \quad \Leftrightarrow \quad 1 - 2\overline{w} - 2w = 0$$

$$\Leftrightarrow \quad 1 - 2(\overline{w} + w) = 1 - 4\mathrm{Re}(w) = 0$$




Therefore, $f(D) = \left\{ z \in \mathbb{C} \mid \mathrm{Re}(z) = \frac{1}{4} \right\} := G.$

Analogously it follows that $f(G) = D$.

The exponential function

$$e^z = e^{x+iy}$$

$$z = x + \mathrm{i}y: \qquad \exp(z) = \mathrm{e}^x \cdot \mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x \overbrace{(\cos(y) + \mathrm{i}\sin(y))}^{2\pi\text{-periodic}}.$$

Then

$$|\mathrm{e}^z| = \mathrm{e}^{\mathrm{Re}(z)}, \qquad \arg(\mathrm{e}^z) = \mathrm{Im}(z) \quad (+2k\pi).$$

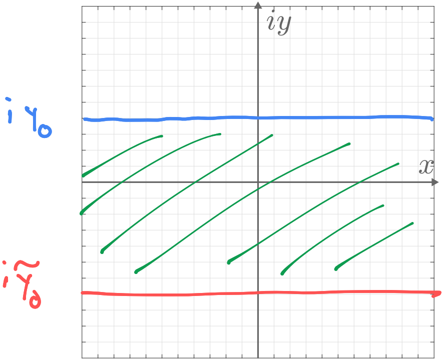
$$|e^z| = |e^x \cdot e^{iy}| = |e^x| \cdot \underbrace{|e^{iy}|}_{=1} = |e^x|$$

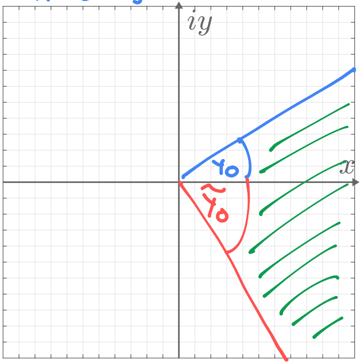
$$e^z = e^x \cdot e^{iy_0}$$

variable radius

fixed angle

**Example:** $z = x + \mathrm{i}y_0$ with $x \in \mathbb{R}$ and fixed $y_0$.




$\boxed{\text{Strips parallel to the real axis are mapped to sectors}}$

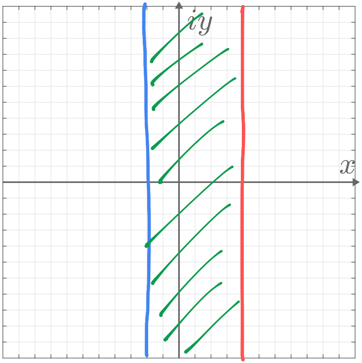
$x \in (-\infty, \infty) \Rightarrow e^x \in (0, \infty)$

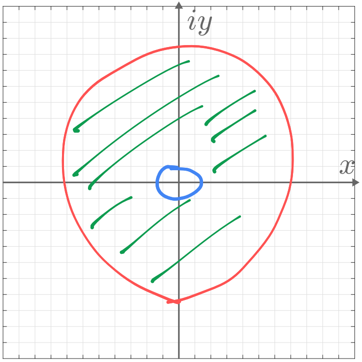
$e^z = e^{x_0} \cdot e^{iy}$

fixed radius

variable angle

**Example:** $z = x_0 + iy$ with $y \in \mathbb{R}$ and fixed $x_0$.




inner radius: $e^{x_0}$

outer radius: $e^{\tilde{x}_0}$

Strips parallel to the imaginary axis are mapped to rings around 0.

# Composition of functions

**Example:** $f(z) = (\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \cdot z)^2 + 1 + \mathrm{i}$,

$$M = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2,\ \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) < 0\}$$

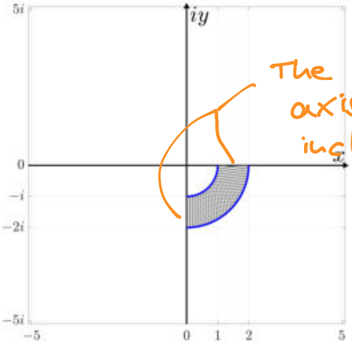
Decompose $f$ into:

$$f = f_3 \circ f_2 \circ f_1$$

with

$$u = f_1(z) = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \cdot z, \qquad v = f_2(u) = v^2, \qquad w = f_3(v) = v + 1 + \mathrm{i}.$$

$$
\begin{aligned}
M &= \{z \in \mathbb{C} \mid 1 \le |z| \le 2,\ \underline{\mathrm{Re}(z) > 0},\ \underline{\mathrm{Im}(z) < 0}\} \\
&= \{z = r\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C} \mid r \in [1,2],\ \varphi \in (-\pi/2, 0)\}
\end{aligned}
$$




The parts on the axis are not included.

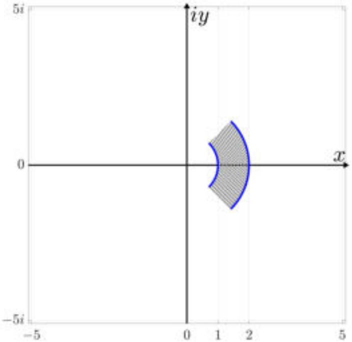
Here: $\rho = 1$

$$u = f_1(z) = \rho\, e^{i\frac{\pi}{4}} \cdot z$$

$\rho \cdot 2$ $\qquad$ $-\frac{\pi}{2} + \frac{\pi}{4}$

$$f_1(M) = \{u = re^{i\varphi} \in \mathbb{C} \mid r \in [1, 2],\ \varphi \in (-\pi/4, \pi/4)\}$$

$\rho \cdot 1$ $\qquad$ $0 + \frac{\pi}{4}$




rotate by $\frac{\pi}{4}$

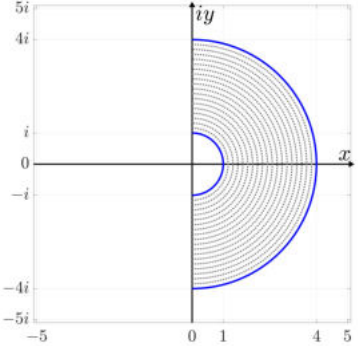
("stretch" by 1.)

$2^2 \qquad 2\cdot\left(-\frac{\pi}{4}\right)$

$$v = f_2(u) = u^2$$

$$f_2(f_1(M)) = \{v = r\mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C} \mid r \in [1, 4],\ \varphi \in (-\pi/2, \pi/2)\}$$

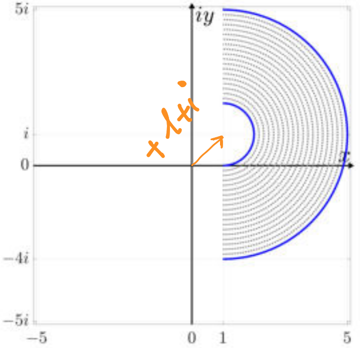
$1^2 \qquad 2\cdot\frac{\pi}{4}$




Square the radii,
double the
angles

$$w = f_3(v) \;=\; v + 1 + \mathrm{i}$$

$$f(M) = f_3(f_2(f_1(M))) \;=\; \{w \in \mathbb{C} \mid w = re^{\mathrm{i}\varphi} \underline{+ 1 + \mathrm{i}},\; r \in [1, 4],\; \varphi \in (-\pi/2, \pi/2)\}$$




Shift by 1+i

We now know some geometric operations in the complex plane.

- Shifting: Addition
- Rotation / stretching: Multiplication
- Increase / decrease sectors: Powers
- Map stripes parallel to the imaginary axis to rings: $\exp$
- Map stripes parallel to the real axis to sectors: $\exp$

# 2. Natural powers / roots

## What is the square root of a complex number?

**We already know:** $z = re^{i\varphi}, \quad n \in \mathbb{N} \quad \Rightarrow \quad z^n = r^n e^{in\varphi}.$

**Question:** Is $\sqrt[n]{z} = z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\varphi}{n}}$ ? $\rightarrow$ **It is a bit more complicated.**

With this we get $\left(r^{\frac{1}{n}} \cdot e^{i\frac{\varphi}{n}}\right)^n = re^{i\varphi}$, but there's more!

We are looking for $\sqrt{i}$, that is, a $w \in \mathbb{C}$ with $w^2 = \mathrm{i}$.

$$w = \rho e^{i\alpha} = \sqrt{i} \quad \Rightarrow \quad w^2 = \mathrm{i} = 1 \cdot e^{i\frac{\pi}{2}}$$

$$w^2 = \rho^2 e^{i2\alpha} \stackrel{!}{=} 1^2 \cdot e^{i\frac{\pi}{2}}$$

$$e^{i2\alpha} = e^{i\frac{\pi}{2}} = e^{i\left(\frac{\pi}{2}+2k\pi\right)}$$

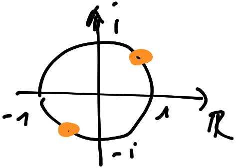
With that:

First find the radius

$$\rho^2 = 1 \quad \Leftrightarrow \quad \rho = 1 \qquad (\text{since} \quad \rho > 0)$$

Then find the angle

$$2\alpha = \frac{\pi}{2} + 2k\pi \quad \Rightarrow \quad \alpha = \frac{\pi}{4} + k\pi, \qquad k \in \mathbb{Z}.$$




Infinitely many angles, but only two points in $\mathbb{C}$.

## General natural roots and their **principal value**

In order for $z = r\mathrm{e}^{\mathrm{i}\varphi}$, $n \in \mathbb{N}$, $w := z^{\frac{1}{n}}$, with $w = \rho\mathrm{e}^{\mathrm{i}\alpha}$, to give

$$w^n = \rho^n \mathrm{e}^{\mathrm{i}n\alpha} = z = |z|\mathrm{e}^{\mathrm{i}\arg(z)},$$

Infinitely many angles

we need $\rho = |z|^{\frac{1}{n}}$ and

$$\mathrm{e}^{\mathrm{i}n\alpha} = \mathrm{e}^{\mathrm{i}\arg(z)} \quad\Rightarrow\quad n\alpha = \arg(z) + 2k\pi \quad\Rightarrow\quad \underline{\alpha = \frac{\arg(z)}{n} + \frac{2k}{n}\pi}$$

This leads to $n$ pairwise different points

Finite number of points

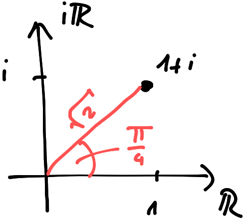
$$w = |z|^{\frac{1}{n}} \exp\left(\mathrm{i}\left(\frac{\arg(z)}{n} + \frac{2k}{n}\pi\right)\right), \qquad \underline{k = 0, 1, 2 \cdots, n-1,}$$

with $w^n = z$. we call $w = |z|^{\frac{1}{n}}\mathrm{e}^{\mathrm{i}\frac{\arg(z)}{n}}$, with $\arg(z) \in (-\pi, \pi)$ the **principal value** of the $n$-th root.

As in $\mathbb{R}$: $\sqrt{4} = 2$, not $-2$.
But $x^2 = 4$ has the solutions $2$ and $-2$.

**Example:** Find all solutions $z \in \mathbb{C}$ of

$$\frac{1}{\sqrt{2}}(1+\mathrm{i})\, z^3 = 27\mathrm{i}$$




$$\frac{1}{\sqrt{2}}(1+i) = e^{i\frac{\pi}{4}}$$

In polar coordinates: $e^{i\frac{\pi}{4}} z^3 = 27 e^{i\frac{\pi}{2}} \Rightarrow z^3 = 27 e^{i\frac{\pi}{4}}$

(1) Find the radius: $|z^3| = |z|^3 \overset{!}{=} \left|27 e^{i\frac{\pi}{4}}\right| = 27$

$$\Rightarrow |z| = \sqrt[3]{27} = 3$$

↳ All solutions have radius 3, $z = 3e^{i\alpha}$.

(2) Find the angle:

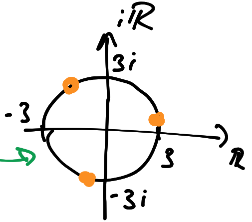
$$z^3 = (3e^{i\alpha})^3 = 27 e^{i3\alpha} \overset{!}{=} 27 e^{i\frac{\pi}{4}}$$

$$\Rightarrow e^{i3\alpha} = e^{i\frac{\pi}{4}} \Rightarrow 3\alpha = \frac{\pi}{4} + 2k\pi$$

Infinitely many angles

$$\Rightarrow \alpha = \frac{\pi}{12} + \frac{2k}{3}\pi$$

Finitely many points




Three solutions: $w_1 = 3e^{i\frac{\pi}{12}}$, $w_2 = 3e^{i\frac{9}{12}\pi}$, $w_3 = 3e^{i\frac{17}{12}\pi}$

Some equations with exponential terms can be solved in a similar fashion.

**Example:** Find all solutions $z \in \mathbb{C}$ of

$$(e^z)^2 = e^{2z} = -9i = 9e^{i(-\frac{\pi}{2})}$$

Similarly: With $z = x + iy$:

(1) Find the real part: Since $|(e^z)^2| = |e^z|^2 = (e^x)^2$:

$|(e^z)^2| \overset{!}{=} |9e^{i(-\pi/2)}| = 9 \Rightarrow (e^x)^2 = 9 \Rightarrow e^x = 3 \Rightarrow x = \ln(3)$

$\rightarrow$ All solutions have real part $\ln(3)$, $z = \ln(3) + iy$.

(2) Find the imaginary part:

$(e^z)^2 = 9e^{i2y} \overset{!}{=} 9e^{i(-\frac{\pi}{2})}$

$\Rightarrow 2y = -\frac{\pi}{2} + 2k\pi \Rightarrow y = -\frac{\pi}{4} + k\pi$

$k \in \mathbb{Z}$

These are indeed infinitely many points!