Universität Hamburg
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Complex functions
for students of engineering sciences

Notes on the exam

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1. Exam Mathematics IV:

Date, place, procedures, materials and tools

Date, place, process

Date: Wed., 27.08.25, 13:00 - 15:00 (KoFu and DGL II)

Place: CU Arena *(S - Neugraben)*
Am Johannisland 2
21147 Hamburg

For **PDEs and Complex Fuctions**:

- You get two exams.
- You have 120 minutes, which you can spend as you prefer.
- In each exam, the maximum number of points is 20.
- In the end, both exams are treated as one big exam.

More on the process

The hall will be separated into **two areas**:

- PDEs and Complex Fuctions (German)
- PDEs and Complex Fuctions (English)

When you're inside the hall:

- Take a seat, stay seated.
- We hand out the exams. You fill out the covers. You **must not open the exam** until you get the signal to do so. **Opening the exams early counts as an attempt to cheat!**

During the exam:

- We check the attendance. Please put a photo-ID and the cover of one exam easily visible on your desk.
- Unfortunately, we will have to briefly interrupt you when we're checking the cover of the exam you're working on.

Allowed materials

You **have to** bring:

- Something to write with (no pencils, no red ink), you do not need your own paper.
- An official photo-ID (passport, driver's licence, etc.)

You **can** bring:

- Anything on paper (text books, lecture notes, your own notes, ...)
- a non-smart watch that doesn't produce distracting sounds

You **must not** bring:

- smart phones, smart watches
- pocket calculators or other electronic devices

2. Exam Complex Functions: Typical topics

Preliminary remarks

In the following we discuss some topics that have typically occurred in previous exams. This should give you some orientation when preparing for the exam.

However:

- If a topic from the lecture / the exercises does not show up in these notes it **does not** follow that it cannot show up on the exam.
- We cannot repeat everything in these notes, there will only be some bullet points.
- In particular, I will not go into all details concerning conditions under which certain statements hold.

Basics: Representation of complex numbers, elementary operations, simple sets

Representation of complex numbers

$z = x + \mathrm{i}y, \quad x, y \in \mathbb{R}$ or $z = r\mathrm{e}^{\mathrm{i}\varphi},\ r > 0,\ \varphi \in [-\pi, \pi)$

Elementary operations

$\mathrm{Re}(z), \quad \mathrm{Im}(z), \quad \overline{z}, \quad |z|, \quad \arg(z)$

Simple subsets of $\mathbb{C}$

Circles, rings, sectors, quadrants, stripes, rectangles

Solutions of complex equations $(\rightarrow$ P2, P3$)$

Roots and logarithms

For $k \in \mathbb{Z}, \quad \arg(z) \in (-\pi, \pi)$:

- $$w^n = z \quad \Rightarrow \quad w = |z|^{1/n} \cdot \exp\left(\mathrm{i}\left(\frac{\arg(z)}{n} + \frac{2k\pi}{n}\right)\right)$$
- $$\mathrm{e}^{nw} = z \quad \Rightarrow \quad w = \ln\left(|z|^{1/n}\right) + \mathrm{i} \cdot \left(\frac{\arg(z)}{n} + \frac{2k\pi}{n}\right)$$

We call $\ln(z) = \ln(|z|) + \mathrm{i} \cdot \arg(z)$, with $\arg(z) \in (-\pi, \pi)$, the **principal values** of the complex logarithm.

Images of elementary functions $(\rightarrow$ P2, HA 2)

Elementary functions

$az + b, \quad z^k, \quad \mathrm{e}^z, \quad \ln(z)$ and their compositions

What are the images of circles, rings, sectors, quadrants, stripes, rectangles, under elementary functions?

The Möbius transformation ($\rightarrow$ P3, P4, HA3, HA4)

Möbius transformation

$$T : \mathbb{C}^* \to \mathbb{C}^*, \qquad T(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0$$

For a Möbius transformation T the following hods:

- T is uniquely determined by three interpolation conditions,
  $$T(z_1) = y_1, \quad T(z_2) = y_2, \quad T(z_3) = y_3.$$
  In particular, for $y_1 = 0,\ y_2 = \infty :\quad T(z) = \alpha \dfrac{z - z_1}{z - z_2}.$
- T maps generalized circles to generalized circles.
- G generalized circle: $-d/c \in G \quad \Leftrightarrow \quad T(G)$ is a straight line.
- T preserves symmetries w.r.t. generalized circles.
- T is conformal in all $z \neq -d/c$.

Complex derivatives ($\to$ P4, HA4)

Cauchy-Riemann-ODEs

$f(z) = u(x,y) + \mathrm{i}v(x,y)$ is complex differentiable in $z = x + \mathrm{i}y$ $\Leftrightarrow$

$$u_x(x,y) = v_y(x,y) \qquad \text{and} \qquad u_y(x,y) = -v_x(x,y).$$

In that case, $f'(z) = u_x(x,y) + \mathrm{i}v_x(x,y)$.

- f diff.able in a neighbourhood of z_0 $\Rightarrow$ u, v harmonic
- For u with $\Delta u = 0$ we can find v with $\Delta v = 0$, such that $f = u + \mathrm{i}v$ is diff.able (**conjugate harmonic function**, potential problem from CR-ODEs).
- f **conformal** (angle preserving) in z_0 $\Leftrightarrow$ $f'(z_0) \neq 0$
- $\exp'(z) = \exp(z)$, $\sin'(z) = \cos(z)$, $(z^n)' = nz^{n-1}, \ldots$
- G: open domain, f diff.able in all $z \in G$: f is called **analytic**.

Complex curve integrals (→ P5, HA5)

$G \subset \mathbb{C}$, Γ: Curve in G, parametrized by $\Gamma = \{c(t) \mid t \in [a,b]\}$, $f : G \to \mathbb{C}$

Complex curve integral

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f\left(c(t)\right) c'(t)\,\mathrm{d}t.$$

- If G **simply connected** and f **analytic** on G with $F' = f$:

$$\int_\Gamma f(z)\,\mathrm{d}z = F(c(b)) - F(c(a))$$

(→ path independence)

- If additionally Γ is **closed**:

$$\int_\Gamma f(z)\,\mathrm{d}z = 0$$

(→ Cauchy integral theorem)

Cauchy integral formulas $(\rightarrow$ P5, HA5$)$

G: simply connected open domain, $z_0 \in G$

Γ: closed curve in G, $\operatorname{Ind}(z_0, \Gamma) = 1$

f : analytic on G

Cauchy integral formulas

- $$\int_\Gamma \frac{f(z)}{(z - z_0)}\,\mathrm{d}z = 2\pi\mathrm{i} \cdot f(z_0)$$
- $$\int_\Gamma \frac{f(z)}{(z - z_0)^n}\,\mathrm{d}z = 2\pi\mathrm{i} \cdot \frac{f^{(n-1)}(z_0)}{(n-1)!}$$

Analytic functions

G : open domain, f analytic on G

Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z - z_0)^n, \qquad z \in B_R(z_0)$$ (largest circle around z_0 in G)

- $$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z$$
- Usually simpler: Use known series expansions.

Laurent series $(\rightarrow$ P6, HA 6)

Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \underbrace{\sum_{n=-\infty}^{-1} c_n (z-z_0)^n}_{principal\ part} + \underbrace{\sum_{n=0}^{\infty} c_n (z-z_0)^n}_{minor\ part}$$

Expansion point $z_0 \in \mathbb{C}$, $\quad z \in R_{r_1}^{r_2} = \{z \in \mathbb{C} \mid r_1 < |z-z_0| < r_2\}$
Converges in the largest ring around z_0 that contains no singularities.

- In different rings we get different Laurent series.
- $c_n = \dfrac{1}{2\pi} \displaystyle\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$
- If f in $B_{r_2}(z_0)$ is analytic, the principal part vanishes.
- If possible, use know series expansions of the function (or of parts of the function). Often we can use the trick with the geometric series.

Isolated singularities $(\to$ P6, P7, HA6, HA 7)

Isolated singularities

f analytic: isolated singularity at z_0, if for some $r > 0$ the function f in defined for all z with $0 < |z - z_0| < r$, but not in z_0.

We distinguish between **removable singularities**, **poles** and **essential singularities**.

- Singularities can be characterized using the principal part of the Laurent series.
- Rational functions never have essential singularities.
- Often it's simpler:
  - z_0 is removable if $\lim_{z\to z_0}(z - z_0)f(z) = 0$.
  - z_0 is a pole of order m, if $\lim_{z\to z_0}(z - z_0)^m f(z)$ exists and is not zero.
  - In particular, for $f(z) = 1/q(z)$, q polynomial: Roots of q are poles with order corresponding to their multiplicity.

Residues $(\rightarrow$ P7, HA 7)

The residue

f analytic, iso. singularity at z_0, Laurent series: $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$

We call $\operatorname{Res}(f, z_0) := c_{-1} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(z)\,\mathrm{d}z$ the **residue** of f at z_0.

Computing residues:

- z_0 removable $\Rightarrow$ $\operatorname{Res}(f, z_0) = 0$
- z_0 simple pole $\Rightarrow$ $\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$
- z_0 pole of order m $\Rightarrow$ $\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \frac{(z - z_0)^m f(z)}{(m-1)!}$
- $f(z) = \frac{p(z)}{q(z)}$, $p,\ q$ analytic witch $q(z_0) = 0,\ q'(z_0) \neq 0$

  $\Rightarrow$ $\operatorname{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$

Residue theorem $(\to$ P7, HA 7)

f: analytic on $G \setminus \{z_1, \ldots, z_n\}$

Γ: closed curve in G, which goes around each z_k with respective index $\mathrm{Ind}(z_k, \Gamma)$

Residue theorem $(\to$ P7, HA 7)

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \cdot \sum_{k=1}^{n} \mathrm{Ind}(z_k, \Gamma) \cdot \mathrm{Res}(f, z_k)$$

$\mathrm{Ind}(z_k, \Gamma) =$ # positive loops $-$ # negative loops

More applications of residues $(\rightarrow$ P7, HA 7)

Improper integrals

$f(z) = \dfrac{p(z)}{q(z)}$, $p,\ q$ polynomials with real coefficients and $\deg(q) \geq \deg(p) + 2$,

no root z_k of q is real $\Rightarrow$ $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i} \cdot \sum_{\mathrm{Im}(z_k)>0} \mathrm{Res}(f, z_k)$

Partial fractions

$f(z) = \dfrac{p(z)}{q(z)}$, $p,\ q$ polynomials with $m = \deg(q) > \deg(p)$,

$z_1, \ \ldots, z_m$ simple roots of q $\Rightarrow$ $f(z) = \dfrac{\mathrm{Res}(f, z_1)}{z - z_1} + \cdots + \dfrac{\mathrm{Res}(f, z_m)}{z - z_m}$

3. Additional remarks

Additional remarks

- The content of the exam will mostly follow the material discussed in the working sheets.
- There are $2 \cdot 20$ points and 120 minutes time, i.e. on average 3 minutes per point. For a problem with three points you should not spend 20 minutes on computations. If a problem has seven points, it is unlikely that you can solve it by one line of computations.

What will **not** be on the exam:

- Fourier transformations
- (complicated) proofs

Additional resources

Old exams:

- You can find old exams on the website for the teaching export (link in Stud.IP). Most of the are in German only, though.
- Note that teachers in previous courses may have put their emphasis on different topics and may have used a different notation.
- The exam this year could also contain different types of problems.

Office hours / contact:

- In the week **18.08. - 22.08.** we will offer additional office hours at UHH (dates will follow on the website for teaching export and in Stud.IP)
- Simple questions can be answered by mail (claus.goetz@uni-hamburg.de or DM in Stud.IP).