Fachbereich Mathematik der Universität Hamburg **SoSe 2025**
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Complex Functions for Students of Engineering Sciences

Homework 7 - Solutions

Problem 1. For $\Gamma = \partial B_\pi(0)$ compute the following curve integrals:

$$\text{(a)} \quad \int_\Gamma \frac{\sin(z)}{z^2}\,\mathrm{d}z, \qquad\qquad \text{(b)} \quad \int_\Gamma \frac{z}{\cos(2z)}\,\mathrm{d}z.$$

Solution.

(a) The only singularity lies at $z_0 = 0$. This is a simple pole, because $zf(z) = \sin(z)/z$ is bounded in a neighbourhood of 0. By a Taylor expansion around $z_0 = 0$ we get for $f(z) = \sin(z)/z^2$:

$$f(z) = \frac{z \; - \; z^3/3! \; + \; z^5/5! \; + \; \ldots}{z} \;=\; \frac{1}{z} - \frac{1}{6}z + \frac{1}{120}z^3 + \ldots.$$

Again, we see that $z_0 = 0$ is a simple pole, because -1 is the smallest power of z in the Laurent series. The corresponding coefficient is

$$c_{-1} = 1 = \mathrm{Res}(f, 0).$$

By the residue theorem we get

$$\int_\Gamma \frac{\sin(z)}{z^2}\,\mathrm{d}z \;=\; 2\pi\mathrm{i}\cdot\mathrm{Res}(f,0) \;=\; 2\pi\mathrm{i}.$$

(b) Let $f(z) = z/\cos(2z)$. The zeros of $\cos(2z)$ are

$$2z = \frac{\pi}{2} + k\pi \qquad \Rightarrow \qquad z = \frac{\pi}{4} + k\frac{\pi}{2} \quad k \in \mathbb{Z}.$$

Of those, only the points

$$z_{1,2} = \pm\frac{3\pi}{4}, \qquad z_{3,4} = \pm\frac{\pi}{4}$$

lie in the interior of $B_\pi(0)$.

With $p(z) = z,\ q(z) = \cos(2z)$ we have $f(z) = p(z)/q(z)$. Since $q(z_k) = 0,\ q'(z_k) \neq 0$, the z_k are simple poles with

$$\mathrm{Res}(f, z_z) \;=\; \frac{p(z_k)}{q'(z_k)} \;=\; -\frac{z_k}{2\sin(2z_k)}.$$

Therefore,

$$\begin{aligned}
\operatorname{Res}\left(f, \frac{\pi}{4}\right) &= -\frac{\pi/4}{2\sin(\pi/2)} &&= -\frac{\pi}{8}, \\
\operatorname{Res}\left(f, -\frac{\pi}{4}\right) &= -\frac{-\pi/4}{2\sin(-\pi/2)} &&= -\frac{\pi}{8}, \\
\operatorname{Res}\left(f, \frac{3\pi}{4}\right) &= -\frac{3\pi/4}{2\sin(3\pi/2)} &&= \frac{3\pi}{8}, \\
\operatorname{Res}\left(f, -\frac{\pi}{4}\right) &= -\frac{-3\pi/4}{2\sin(-3\pi/2)} &&= \frac{3\pi}{8}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_\Gamma \frac{z}{\cos(2z)}\,\mathrm{d}z &= 2\pi\mathrm{i}\cdot\left(\operatorname{Res}\left(f, \frac{\pi}{4}\right) + \operatorname{Res}\left(f, -\frac{\pi}{4}\right) + \operatorname{Res}\left(f, \frac{3\pi}{4}\right) + \operatorname{Res}\left(f, -\frac{\pi}{4}\right)\right) \\
&= \pi^2\mathrm{i}.
\end{aligned}$$

Problem 2.

(a) Let $t \in \mathbb{R}$ and $z = \mathrm{e}^{\mathrm{i}t}$. Show that $\cos(t) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ gilt.

(b) Let $B_1(0)$ denote the open unit disk and let $a \in B_1(0) \setminus \{0\}$. Show that

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos(t) + a^2}\,\mathrm{d}t = \int_{\partial B_1(0)} \frac{\mathrm{i}/a}{(z-a)(z-1/a)}\,\mathrm{d}z.$$

(c) Show that

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos(t) + a^2}\,\mathrm{d}t = \frac{2\pi}{1-a^2}.$$

Solution.

(a) We compute directly

$$\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}\right) = \frac{1}{2}\left(\cos(t) + \mathrm{i}\sin(t) \;+\; \cos(-t) + \mathrm{i}\sin(-t)\right) = \cos(t).$$

(b) Consider $c(t) = \mathrm{e}^{\mathrm{i}t},\ t \in [0, 2\pi)$ and

$$f(z) = \frac{\mathrm{i}/a}{(z-a)(z-1/a)}.$$

With this we find

$$\begin{aligned}
f(c(t))c'(t) &= \frac{\mathrm{i}/a}{(\mathrm{e}^{\mathrm{i}t}-a)(\mathrm{e}^{\mathrm{i}t}-1/a)}\cdot \mathrm{i}\mathrm{e}^{\mathrm{i}t} = -\frac{1}{a}\cdot\frac{1}{(\mathrm{e}^{\mathrm{i}t})^2 - a\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}t}/a + 1}\cdot \mathrm{e}^{\mathrm{i}t} \\
&= \frac{1}{1 + a(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}) + a^2} = \frac{1}{1 - 2a\cos(t) + a^2}.
\end{aligned}$$

Therefore,

$$\int_{\partial B_1(0)} \frac{\mathrm{i}/a}{(z-a)(z-1/a)}\,\mathrm{d}z = \int_0^{2\pi} f(c(t))c'(t)\,\mathrm{d}t = \int_0^{2\pi} \frac{1}{1-2a\cos(t)+a^2}\,\mathrm{d}t.$$

(c) The only singularity of f in $B_1(0)$ lies at $z_0 = a$. We compute

$$\mathrm{Res}(f,a) \;=\; \left.\frac{\mathrm{i}/a}{z-1/a}\right|_{z=a} \;=\; \frac{\mathrm{i}}{a^2-1}.$$

From the residue theorem it follows that

$$\int_0^{2\pi} \frac{1}{1-2a\cos(t)+a^2}\,\mathrm{d}t \;=\; \int_{\partial B_1(0)} \frac{\mathrm{i}/a}{(z-a)(z-1/a)}\,\mathrm{d}z \;=\; 2\pi\mathrm{i}\mathrm{Res}(f,a) \;=\; \frac{2\pi}{1-a^2}.$$