Fachbereich Mathematik der Universität Hamburg **SoSe 2025**
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Complex Functions for Students of Engineering Sciences

Homework 6 - Solutions

Problem 1. Let ln denote the principal branch of the natural logarithm and consider

$$f(z) \;=\; \ln(1+z) \;+\; \ln(1-z).$$

(a) Show that for $n \in \mathbb{N}$ the n-th derivative of f is given by

$$f^{(n)}(z) = -(n-1)!\left(\frac{(-1)^n}{(1+z)^n} + \frac{1}{(1-z)^n}\right).$$

(b) Compute the Taylor series of f around $z_0 = \mathrm{i}$ and determine its radius of convergence.

Solution.

(a) Of course, we can compute this directly. But we show the claim by induction for n:

Base, $n = 1$:

$$f'(z) \;=\; \frac{1}{1+z} - \frac{1}{1-z} \;=\; -\,(1-1)!\left(\frac{(-1)^1}{(1+z)^1} + \frac{1}{(1-z)^1}\right).$$

Proposition: For some $n \in \mathbb{N}$ it holds

$$f^{(n)}(z) = -(n-1)!\left(\frac{(-1)^n}{(1+z)^n} + \frac{1}{(1-z)^n}\right).$$

Step: $n \to n+1$

$$\begin{aligned}
f^{(n+1)}(z) \;&=\; \frac{\mathrm{d}}{\mathrm{d}z} f^{(n)} \;=\; \frac{\mathrm{d}}{\mathrm{d}z}\left(-(n-1)!\left(\frac{(-1)^n}{(1+z)^n} + \frac{1}{(1-z)^n}\right)\right) \\
&=\; -\,(n-1)!\left(\frac{-n\cdot(-1)^n}{(1+z)^{n+1}} + \frac{-n\cdot(-1)}{(1-z)^{n+1}}\right) \\
&=\; -\,n!\left(\frac{(-1)^{n+1}}{(1+z)^{n+1}} + \frac{1}{(1-z)^{n+1}}\right).
\end{aligned}$$

(b) We have

$$1 \pm \mathrm{i} \;=\; \sqrt{2}\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{4}}$$

and therefore

$$c_0 \;=\; f(\mathrm{i}) \;=\; \ln(\sqrt{2}) + \mathrm{i}\cdot\frac{\pi}{4} \;+\; \ln(\sqrt{2}) - \mathrm{i}\cdot\frac{\pi}{4} \;=\; 2\ln(\sqrt{2}) = \ln(2).$$

For $n \in \mathbb{N}$ we get

$$\begin{aligned} c_n \;&=\frac{f^{(n)}(\mathrm{i})}{n!} \;=\; -\frac{1}{n}\left(\frac{\mathrm{e}^{\mathrm{i}\cdot n\pi}}{(\sqrt{2})^n\mathrm{e}^{\mathrm{i}\cdot n\pi/4}} + \frac{1}{(\sqrt{2})^n\mathrm{e}^{-\mathrm{i}\cdot n\pi/4}}\right) \\ &=\; -\frac{\mathrm{e}^{\mathrm{i}\cdot n\pi/2}}{n\cdot(\sqrt{2})^n}\left(\mathrm{e}^{\mathrm{i}\cdot n\pi/4} + \mathrm{e}^{-\mathrm{i}\cdot n\pi/4}\right) \;=\; -\frac{2}{n}\cdot\left(\frac{\mathrm{i}}{\sqrt{2}}\right)^n\cdot\cos\left(\frac{n\pi}{4}\right). \end{aligned}$$

The Taylor series is

$$f(z) \;=\; \ln(2) + \sum_{n=1}^{\infty} -\frac{2}{n\cdot n!}\cdot\left(\frac{\mathrm{i}}{\sqrt{2}}\right)^n\cdot\cos\left(\frac{n\pi}{4}\right)\cdot(z-\mathrm{i})^n.$$

The series converges in the largest disk aroud $z_0 = \mathrm{i}$ that contains no singularity of f. The nearest singularities of f lie at ± 1, so the radius of convergence is

$$R = |\pm 1 - \mathrm{i}| = \sqrt{2}.$$

Problem 2. Find *all* power series expansions of

$$f(z) = \frac{5z}{z^2 + z - 6}$$

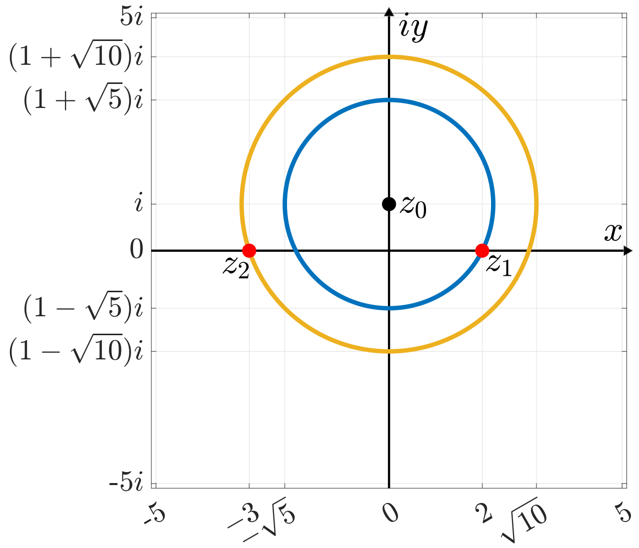
around $z_0 = \mathrm{i}$. Where do these series converge, respectively?

Solution: Factoring the denominator $z^2 + z - 6 = (z-2)(z+3)$ we find the singularities at $z_1 = 2$ and $z_2 = -3$.

Decomposing into partial fractions we get

$$f(z) = \frac{5z}{z^2 + z - 6} = \frac{3}{z+3} + \frac{2}{z-2}.$$

We are expanding around $z_0 = \mathrm{i}$, so from the position of the singularities at $z_1 = 2$ and $z_2 = -3$ we can see we will have a Taylor expansion inside the disk $|z - \mathrm{i}| < \sqrt{5}$, a Laurent expansion in the ring $\sqrt{5} < |z - \mathrm{i}| < \sqrt{10}$ and another different Laurent expansion in the exterior $\sqrt{10} < |z - i|$.



We use the trick with the geometric series to expand the partial fractions:

For $\quad |z-\mathrm{i}| < \sqrt{5}$:

$$\begin{aligned}\frac{2}{z-2} &= \frac{2}{-2+\mathrm{i}+(z-\mathrm{i})} = \frac{2}{-2+\mathrm{i}} \cdot \frac{1}{1-(z-\mathrm{i})/(2-\mathrm{i})} \\ &= \frac{2}{-2+\mathrm{i}} \sum_{n=0}^{\infty} \frac{(z-\mathrm{i})^n}{(2-\mathrm{i})^n} = \sum_{n=0}^{\infty} \frac{-2}{(2-\mathrm{i})^{n+1}} (z-\mathrm{i})^n.\end{aligned}$$

For $\quad |z-\mathrm{i}| > \sqrt{5}$:

$$\begin{aligned}\frac{2}{z-2} &= \frac{2}{-2+\mathrm{i}+(z-\mathrm{i})} = \frac{2}{z-\mathrm{i}} \cdot \frac{1}{1-(2-\mathrm{i})/(z-\mathrm{i})} \\ &= \frac{2}{z-\mathrm{i}} \sum_{n=0}^{\infty} \frac{(2-\mathrm{i})^n}{(z-\mathrm{i})^n} = \sum_{n=-\infty}^{-1} \frac{2}{(2-\mathrm{i})^{n+1}} (z-\mathrm{i})^n.\end{aligned}$$

For $\quad |z-\mathrm{i}| < \sqrt{10}$:

$$\begin{aligned}\frac{3}{z+3} &= \frac{3}{3+\mathrm{i}+(z-\mathrm{i})} = \frac{3}{3+\mathrm{i}} \cdot \frac{1}{1+(z-\mathrm{i})/(3+\mathrm{i})} \\ &= \frac{3}{3+\mathrm{i}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(3+\mathrm{i})^n} (z-\mathrm{i})^n = \sum_{n=0}^{\infty} \frac{-3}{(-3-\mathrm{i})^{n+1}} (z-\mathrm{i})^n.\end{aligned}$$

For $\quad |z-\mathrm{i}| > \sqrt{10}$:

$$\begin{aligned}\frac{3}{z+3} &= \frac{3}{3+\mathrm{i}+(z-\mathrm{i})} = \frac{3}{z-i} \cdot \frac{1}{1+(3+i)/(z-i)} \\ &= \frac{3}{z-i} \sum_{n=0}^{\infty} \frac{(-1)^n (3+\mathrm{i})^n}{(z-\mathrm{i})^n} = \sum_{n=-\infty}^{-1} \frac{3}{(-3-\mathrm{i})^{n+1}} (z-\mathrm{i})^n.\end{aligned}$$

Taylor series converging inside the disk $|z - \mathrm{i}| < \sqrt{5}$:

$$f(z) = \frac{3}{z+3} + \frac{2}{z-2} = \underbrace{\sum_{n=0}^{\infty} \left(\frac{-2}{(2-i)^{n+1}} + \frac{-3}{(-3-i)^{n+1}} \right) (z-i)^n}_{\text{minor part}} .$$

Laurent series converging inside the ring $\sqrt{5} < |z - i| < \sqrt{10}$:

$$f(z) = \frac{3}{z+3} + \frac{2}{z-2} = \underbrace{\sum_{n=-\infty}^{-1} \frac{2}{(2-\mathrm{i})^{n+1}} (z-\mathrm{i})^n}_{\text{principal part}} + \underbrace{\sum_{n=0}^{\infty} \frac{-3}{(-3-\mathrm{i})^{n+1}} (z-\mathrm{i})^n}_{\text{minor part}} .$$

Laurent series converging inside the exterior $\sqrt{10} < |z - \mathrm{i}|$:

$$f(z) = \frac{3}{z+3} + \frac{2}{z-2} = \underbrace{\sum_{n=-\infty}^{-1} \left(\frac{2}{(2-\mathrm{i})^{n+1}} + \frac{3}{(-3-\mathrm{i})^{n+1}} \right) (z-\mathrm{i})^n}_{\text{principal part}} .$$

Problem 3. For a given $R > 0$ let $f : B_R(0) \to \mathbb{C}$ be an analytic function with $f(0) = 0$. Here, $B_R(0) \subset \mathbb{C}$ denotes the open disk around zero with radius $R > 0$.

Consider the function

$$g(z) \;:=\; \begin{cases} \dfrac{f(z)}{z}, & \text{For } \; z \neq 0, \\ f'(0), & \text{For } \; z = 0. \end{cases}$$

Clearly, we have $f(z) = z \cdot g(z)$ for all $z \in B_R(0)$.

Show that g is analytic. Show that g that the Taylor series of g around zero is given by

$$g(z) \;=\; \sum_{n=1}^{\infty} c_n z^{n-1},$$

where $c_n,\; n \in \mathbb{N}_0$ are the coefficients of the Taylor series of f around zero.

Solution. Let f be analytic with $f(0) = 0$. Inside $B_R(0) \setminus \{0\}$ the function g is the quotient of analytic functions and thus itself analytic. The question is: What happens at $z = 0$?

Because $f(0) = 0$, in the Taylor series around zero,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

we have $c_0 = 0$. This series converges for all $z \in B_R(0)$ and because

$$\sum_{n=1}^{\infty} c_n z^n \;=\; z \sum_{n=1}^{\infty} c_n z^{n-1}$$

also the power series $\tilde{g}(z) = \sum_{n=1}^{\infty} c_n z^{n-1}$ converges for all $z \in B_R(0)$. For all $z \in B_R(0)\setminus\{0\}$ we then have $\tilde{g}(z) = g(z)$.

Moreover,

$$\tilde{g}(0) = c_1 = f'(0) = g(0),$$

so g is given on the entirety of $B_R(0)$ by a convergent power series and therefore analytic. Because the series representation is unique, this means that this is the Taylor series of g.