Fachbereich Mathematik der Universität Hamburg **SoSe 2025**

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Complex functions for students of engineering sciences

Homework 2

Problem 1. Which of the following claims are true, which are false? Give a (short) explanation for your answers.

(a) Claim: *Straight lines* in the complex plane of the form

$$G = \{z \in \mathbb{C} \ | \ z = \alpha z_0, \ \alpha \in \mathbb{R}\} \qquad \text{with a fixed } z_0 \in \mathbb{C}$$

are mapped by $f : \mathbb{C} \to \mathbb{C}, \quad f(z) = z^2$, to *beams* of the form

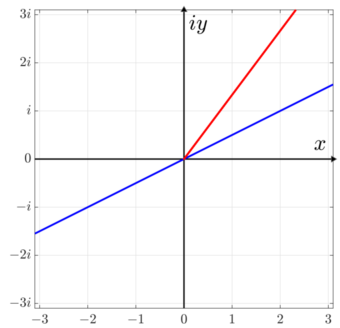
$$H = \{z \in \mathbb{C} \ | \ z = \beta w_0, \ \beta \geq 0 \ \} \qquad \text{with a suitable } w_0 \in \mathbb{C}.$$

(b) Claim: *Circles* in the complex plane with center $z_0 \in \mathbb{C}$ and radius $R > 0$ are mapped by $f : \mathbb{C} \to \mathbb{C}, \quad f(z) = z^2$, to circles with center z_0^2 and radius R^2.

Solution.

(a) The claim is *true*.

Since all $z \in G$ have the form $z = \alpha z_0$, all $w \in f(G)$ have the form $w = z^2 = \alpha^2 z_0^2$, i.e. with $\beta = \alpha^2$ and $w_0 = z_0^2$ the claim follows.



Blue: Line G with $z_0 = 2 + \mathrm{i}$, red: $H = f(G)$

(b) In general, the claim is *wrong*. It is, however, true fro the special case $z_0 = 0$.

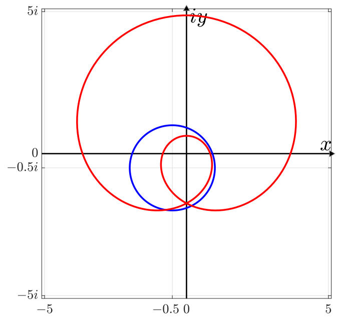
We can express the circle with center $z_0 \in \mathbb{C}$ and radius $R > 0$ as

$$K = \{z \in \mathbb{C} \mid z = z_0 + R\mathrm{e}^{\mathrm{i}\varphi},\ \varphi \in [0, 2\pi)\}.$$

Therefore,

$$f(K) = \{z \in \mathbb{C} \mid z = z_0^2 + R^2\mathrm{e}^{\mathrm{i}2\varphi} + 2z_0R\mathrm{e}^{\mathrm{i}\varphi},\ \varphi \in [0, 2\pi)\}.$$

For $z_0 = 0$ this is a circle around zero with radius R^2 (in which each element has two pre-images in K), but for $z_0 \neq 0$ the term $2z_0R\mathrm{e}^{\mathrm{i}\varphi}$ means that $f(K)$ in general is not a circle.



Blue: Circle K with center $z_0 = -0.5 - 0.5\mathrm{i}$, red: $f(K)$

Problem 2.

(a) Can the set

$$S = \{z \in \mathbb{C} \mid |\mathrm{Re}(z)| + |\mathrm{Im}(z)| = 1\}$$

be mapped to the set

$$Q = \{z \in \mathbb{C} \mid \max\{|\mathrm{Re}(z)|,\ |\mathrm{Im}(z)|\} = 1\}$$

by a *linear transformation*? That is, is there a linear map $f : \mathbb{C} \to \mathbb{C}$, such that $Q = f(S)$?

(b) Can the rectangle with vertices

$$z_1 = 1 + \mathrm{i}\sqrt{3}, \quad z_2 = 1 - \mathrm{i}\sqrt{3}, \quad z_3 = -1 - \mathrm{i}\sqrt{3}, \quad z_4 = -1 + \mathrm{i}\sqrt{3}$$

be mapped to the rectangle with vertices

$$w_1 = 1 + \mathrm{i}, \quad w_2 = 1 - \mathrm{i}, \quad w_3 = -1 - \mathrm{i}, \quad w_4 = -1 + \mathrm{i}$$

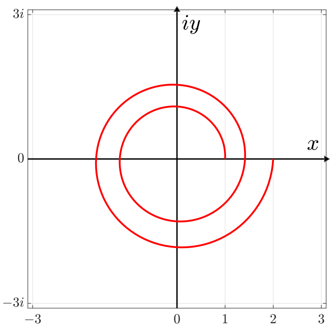
by a linear transformation?

(c) Consider the map $f : \mathbb{C} \to \mathbb{C},\ f(z) = \exp(z)$.

We are looking for a set of the form

$$M = \{z \in \mathbb{C} \mid z = \alpha z_0,\ \alpha \in [a, b]\}$$

for some $z_0 \in \mathbb{C}$ and some interval $[a, b] \subset \mathbb{R}$, such that the image $\exp(M)$ is a spiral, winding twice around zero (in mathematically positive direction), with starting point 1 and end point 2 (see figure on the right). Find a suitable $z_0 \in \mathbb{C}$ and a suitable $[a, b] \subset \mathbb{R}$ for this.

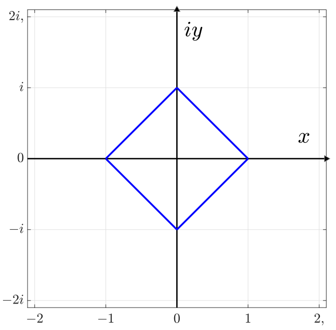


$\exp(M)$

Solution.

(a) The set S is a square with vertices $1, \mathrm{i}, -1, -\mathrm{i}$ (length of the edges: $\sqrt{2}$). Q is a square with edges $1+\mathrm{i},\ 1-\mathrm{i},\ -1-\mathrm{i},\ -1+\mathrm{i}$ (length of the edges: 2). The transformation can be done by rotating and stretching.

More precisely: To get from S to Q we first rotate by $\pi/4$ and then stretch by a factor of $\sqrt{2}$. This is done by the linear map $f(z) = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \cdot z$.

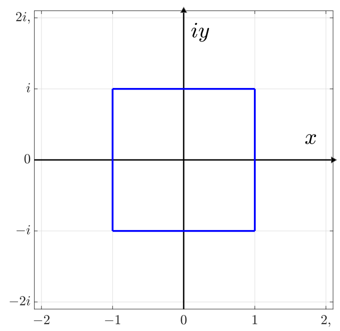


S

$f(z) = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \cdot z$

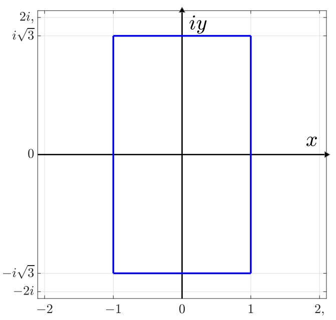
$\longrightarrow$

Rotation and stretching



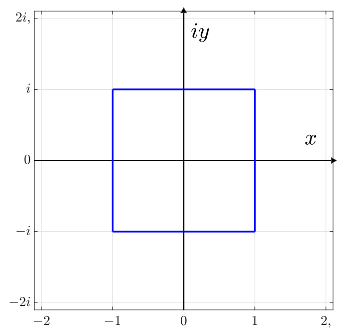
$Q = f(S)$

(b) The rectangle with vertices $z_1, \ldots, z_4$ has two edges with length 2, and two edged with length $2\sqrt{3}$. The vertices $w_1, \ldots, w_4$ define a square. With a linear map, all edges would be stretched by the same factor, so we cannot map this rectangle to a square by a linear map.



Rectangle with edges of length 2 *and* $2\sqrt{3}$

Anisotropic stretching is not possible with a linear map $f : \mathbb{C} \to \mathbb{C}$.



Square with edges of length 2

(c) If we write $z_0 = x_0 + \mathrm{i}y_0$, then

$$\exp(\alpha z_0) = \exp(\alpha x_0) \cdot \exp(\mathrm{i}\alpha y_0).$$

We get a twofold winding with starting point end end point on the positive real half-axis, if αy_0 goes from 0 to 4π, e.g. for $\alpha \in [0, 4\pi]$ and $y_0 = 1$. This also guarantees that the starting point is at $\exp(0 \cdot x_0) = 1$. in order to have the endpoint at 2, we then need

$$\exp(4\pi \cdot x_0) = 2 \quad \Rightarrow \quad x_0 = \frac{\ln(2)}{4\pi}.$$

So we can choose

$$z_0 = \frac{\ln(2)}{4\pi} + \mathrm{i}, \qquad [a, b] = [0, 4\pi].$$

Problem 3. Consider the equation

$$(z-4)^{20} = z^{20}, \qquad z \in \mathbb{C}.$$

Show that this equation has 19 solutions. Furthermore, show that all of these solutions have real part $\mathrm{Re}(z) = 2$.

Solution. We can write

$$(z-4)^{20} \;=\; z^{20} + q(z),$$

where q is polynomial of degree 19. The equation $(z-4)^{20} = z^{20}$ is then equivalent to $q(z) = 0$. Since q has degree 19, it has 19 zeros in the complex plane.

Moreover:

$$(z-4)^{20} \;=\; z^{20} \qquad \Rightarrow \qquad |(z-4)^{20}| \;=\; |z^{20}| \qquad \Rightarrow \qquad |(z-4)|^{20} \;=\; |z|^{20},$$

i.e., each solution has the same distance to 0 as to 4. Therefore, all solutions have real part $\mathrm{Re}(z) = 2$.