Fachbereich Mathematik der Universität Hamburg **SoSe 2025**
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Complex Functions for students of engineering sciences

Homework 1 - Solutions

Using the *Euler formula* we can represent cosine and sine as

$$\cos(x) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}\right), \qquad \sin(x) = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}x} - e^{-\mathrm{i}x}\right) \qquad \text{for} \quad x \in \mathbb{R}.$$

Use these representations to proof the following.

Problem 1: Show that for all $x \in \mathbb{R}$

$$\cos^3(x) = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x).$$

Solution. With the above representation we can write

$$\begin{aligned} \cos^3(x) &= \left(\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}\right)\right)^3 = \frac{1}{8}\left(\mathrm{e}^{3\mathrm{i}x} + 3\mathrm{e}^{\mathrm{i}x} + 3\mathrm{e}^{-\mathrm{i}x} + \mathrm{e}^{-3\mathrm{i}x}\right) \\ &= \frac{1}{4}\frac{\mathrm{e}^{3\mathrm{i}x} + \mathrm{e}^{-3\mathrm{i}x}}{2} + \frac{3}{4}\frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2} \\ &= \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x). \end{aligned}$$

Problem 2:

(a) For some $n \in \mathbb{N}$, let $a_0, \ldots, a_n$ and $b_1, \ldots, b_n$ be given real numbers. We consider the function $f_n : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$f_n(x) := \frac{a_0}{2} + \sum_{\ell=1}^{n} \left[a_\ell \cos(\ell x) + b_\ell \sin(\ell x)\right] \qquad \text{for} \quad x \in \mathbb{R}. \tag{$*$}$$

Show that the function f_n can be expressed as

$$f_n(x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx} \qquad \text{for all } x \in \mathbb{R},$$

with *complex* coefficients $c_{-n}, \ldots, c_n \in \mathbb{C}$. Determine these coefficients

(b) Now let complex numbers $c_{-n}, \ldots, c_n \in \mathbb{C},\ n \in \mathbb{N}$ be given and consider the function $g_n : \mathbb{R} \longrightarrow \mathbb{C}$, defined by

$$g_n(x) := \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx} \qquad \text{for } x \in \mathbb{R}.$$

We assume the following:

(i) $c_0 \in \mathbb{R}$;

(ii) $c_\ell = \overline{c_{-\ell}}$, for $\ell = 1, \ldots, n$. I.e., c_ℓ is the complex conjugate of $c_{-\ell}$.

Show that g_n is real and can be expressed in the form $(*)$ with real coefficients $a_0, \ldots, a_n$ and $b_1, \ldots, b_n$.

Solution.

(a) We use the Euler formula to write

$$\begin{aligned} f_n(x) &= \frac{a_0}{2} + \sum_{\ell=1}^{n} \left[\frac{a_\ell}{2} \left(\mathrm{e}^{\mathrm{i}\ell x} + e^{-\mathrm{i}\ell x} \right) + \frac{b_\ell}{2\mathrm{i}} \left(\mathrm{e}^{\mathrm{i}\ell x} - e^{-\mathrm{i}\ell x} \right) \right] \\ &= \frac{a_0}{2} + \sum_{\ell=1}^{n} \left[\frac{a_\ell - \mathrm{i}b_\ell}{2} \mathrm{e}^{\mathrm{i}\ell x} + \frac{a_\ell + \mathrm{i}b_\ell}{2} e^{-\mathrm{i}\ell x} \right] \\ &= c_0 + \sum_{\ell=1}^{n} \left[c_\ell \mathrm{e}^{\mathrm{i}\ell x} + c_{-\ell} e^{-\mathrm{i}\ell x} \right], \end{aligned}$$

with

$$c_0 = \frac{a_0}{2}, \qquad c_\ell = \frac{a_\ell - \mathrm{i}b_\ell}{2}, \qquad c_{-\ell} = \frac{a_\ell + \mathrm{i}b_\ell}{2}, \qquad \ell = 1, \ldots, n.$$

So we have

$$f_n(x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx} \qquad \text{for all } x \in \mathbb{R}.$$

(b) At first, we write

$$g_n(x) = c_0 + \sum_{\ell=11}^{n} \left[(\mathrm{Re}(c_\ell) + \mathrm{i}\mathrm{Im}(c_\ell))\, \mathrm{e}^{\mathrm{i}jx} \; + \; (\mathrm{Re}(c_{-\ell}) + \mathrm{i}\mathrm{Im}(c_{-\ell}))\, \mathrm{e}^{-\mathrm{i}jx} \right].$$

Since $c_\ell = \overline{c_{-\ell}}$ we have

$$\mathrm{Re}(c_{-\ell}) = \mathrm{Re}(c_\ell) = \frac{1}{2}(c_\ell + c_{-\ell}), \qquad -\mathrm{Im}(c_{-\ell}) = \mathrm{Im}(c_\ell) = \frac{1}{2\mathrm{i}}(c_\ell - c_{-\ell}).$$

Therefore, we can write g_n as

$$g_n(x) = c_0 + \sum_{\ell=1}^{n} \left[(c_\ell + c_{-\ell})\, \frac{\mathrm{e}^{\mathrm{i}\ell x} + \mathrm{e}^{-\mathrm{i}\ell x}}{2} \; + \; \mathrm{i}\,(c_\ell - c_{-\ell})\, \frac{\mathrm{e}^{\mathrm{i}\ell x} - \mathrm{e}^{-\mathrm{i}\ell x}}{2\mathrm{i}} \right].$$

With

$$a_0 = 2c_0, \qquad a_\ell = (c_\ell + c_{-\ell}), \qquad b_\ell = \mathrm{i}\,(c_\ell - c_{-\ell}), \quad \ell = 1, \ldots, n,$$

the function g_n can be written in the form $(*)$. Note that a_ℓ, b_ℓ are indeed real numbers!