Fachbereich Mathematik der Universität Hamburg **SoSe 2025**
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Complex Functions for Students of Engineering Sciences

Worksheet 7 - Solutions

Problem 1. Consider the functions

$$(1) \qquad f(z) \;=\; \frac{1}{z^2-2z+3}, \qquad\qquad (2) \qquad g(z) \;=\; \frac{1}{z^4+3\mathrm{i}z^3-2z^2}.$$

(a) Determine and classify all isolated singularities of f and g, respectively.

(b) Compute the corresponding residues.

(c) Find a partial fraction decomposition for f and g, respectively.

(d) Compute $\oint_\Gamma f(z)\,\mathrm{d}z$ and $\oint_\Gamma g(z)\,\mathrm{d}z$ for $\Gamma = \partial B_2(\mathrm{i}/2)$.

Solution.

(a) We have

$$z^2-2z+3 = (z-1)^2+2 = 0 \qquad z^f_{1,2} = 1 \pm \sqrt{2}\mathrm{i}.$$

At z^f_1 and z^f_2 lie simple poles of f.

Moreover:

$$z^4+3\mathrm{i}z^3-2z^2 = z^2(z+\mathrm{i})(z+2\mathrm{i}) \qquad z^g_0 = 0, \quad z^g_1 = -\mathrm{i}, \quad z^g_2 = -2\mathrm{i}.$$

At z^g_0 lies a double pole of g, at z^g_1, z^g_2 we have simple poles.

(b) Since z^f_1, z^f_2 are simple poles of f, we get:

$$\operatorname{Res}(f, 1+\sqrt{2}\mathrm{i}) \;=\; \lim_{z\to 1+\sqrt{2}\mathrm{i}} (z-(1+\sqrt{2}\mathrm{i}))f(z) \;=\; \left.\frac{1}{z-(1-\sqrt{2}\mathrm{i})}\right|_{z=1+\sqrt{2}\mathrm{i}} \;=\; \frac{1}{2\sqrt{2}\mathrm{i}},$$

$$\operatorname{Res}(f, 1-\sqrt{2}\mathrm{i}) \;=\; \lim_{z\to 1-\sqrt{2}\mathrm{i}} (z-(1-\sqrt{2}\mathrm{i}))f(z) \;=\; \left.\frac{1}{z-(1+\sqrt{2}\mathrm{i})}\right|_{z=1-\sqrt{2}\mathrm{i}} \;=\; -\frac{1}{2\sqrt{2}\mathrm{i}}.$$

Alternatively, we can compute the residues as

$$\operatorname{Res}(f, 1\pm\sqrt{2}\mathrm{i}) \;=\; \left.\frac{1}{(z^2-2z+3)'}\right|_{1\pm\sqrt{2}} \;=\; \left.\frac{1}{2z-2}\right|_{1\pm\sqrt{2}} \;=\; \frac{1}{\pm\sqrt{2}\mathrm{i}}.$$

For z_1^g, z_2^g we get analogously:

$$\operatorname{Res}(g,-\mathrm{i}) \;=\; \lim_{z\to-\mathrm{i}}(z+\mathrm{i})g(z) \;=\; \left.\frac{1}{z^2(z+2\mathrm{i})}\right|_{z=-\mathrm{i}} \;=\; -\frac{1}{\mathrm{i}} \;=\; \mathrm{i},$$

$$\operatorname{Res}(g,-2\mathrm{i}) \;=\; \lim_{z\to-2\mathrm{i}}(z+2\mathrm{i})g(z) \;=\; \left.\frac{1}{z^2(z+\mathrm{i})}\right|_{z=-2\mathrm{i}} \;=\; \frac{1}{4\mathrm{i}} \;=\; -\frac{\mathrm{i}}{4}.$$

Because $z_0^g = 0$ is a pole of order $m = 2$, it holds

$$\operatorname{Res}(g,0) \;=\; \lim_{z\to0}\left(\frac{\mathrm{d}^{(m-1)}}{\mathrm{d}z^{(m-1)}}z^2g(z)\right) \;=\; \lim_{z\to0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z^2+3\mathrm{i}z-2}\right)$$

$$= \; -\left.\frac{2z+3\mathrm{i}}{(z^2+3\mathrm{i}z-2)^2}\right|_{z=0} \;=\; -\frac{3}{4}\mathrm{i}.$$

(c) The function f has two simple poles. The partial fraction decomposition is given by

$$f(z) \;=\; h_f(z;1+\sqrt{2}\mathrm{i}) \;+\; h_f(z;1-\sqrt{2}\mathrm{i}) \;=\; \frac{\operatorname{Res}(f,1+\sqrt{2}\mathrm{i})}{z-(1+\sqrt{2}\mathrm{i})} \;+\; \frac{\operatorname{Res}(f,1-\sqrt{2}\mathrm{i})}{z-(1-\sqrt{2}\mathrm{i})}$$

$$= \; \frac{\mathrm{i}}{2\sqrt{2}}\cdot\frac{1}{z-(1-\sqrt{2}\mathrm{i})} \;-\; \frac{\mathrm{i}}{2\sqrt{2}}\cdot\frac{1}{z-(1+\sqrt{2}\mathrm{i})}.$$

For g we find

$$g(z) \;=\; h_g(z;0) \;+\; h_g(z;-i) \;+\; h_g(z;-2\mathrm{i}),$$

where the last two terms are given as:

$$h_g(z;-i) = \frac{\operatorname{Res}(g,-\mathrm{i})}{z+\mathrm{i}} = \frac{\mathrm{i}}{z+\mathrm{i}}, \qquad h_g(z;-2i) = \frac{\operatorname{Res}(g,-2\mathrm{i})}{z+2\mathrm{i}} = -\frac{\mathrm{i}}{4(z+2\mathrm{i})}.$$

For remaining term, we look at the Taylor series of $\tilde{g}(z) = z^2g(z)$ around $z_0 = 0$:

$$\tilde{g}(z) \;=\; \frac{1}{z^2+3\mathrm{i}z-2} \;=\; -\frac{1}{2} - \frac{3\mathrm{i}}{4}z + \ldots.$$

From this it follows that the Laurent series of $g = \tilde{g}/z^2$ is:

$$g(z) = \underbrace{-\frac{1}{2z^2} - \frac{3\mathrm{i}}{4z}}_{h_g(z,0)} + \ldots.$$

Finally, we get

$$g(z) \;=\; \frac{\mathrm{i}}{z+\mathrm{i}} \;-\; \frac{\mathrm{i}}{4(z+2\mathrm{i})} - \frac{1}{2z^2} - \frac{3\mathrm{i}}{4z}.$$

(d) For f, only the singularity $z_1^f = 1+\sqrt{2}\mathrm{i}$ lies in the interior of $B_2(\mathrm{i}/2)$. With that:

$$\oint_\Gamma f(z)\;\mathrm{d}z \;=\; 2\pi\mathrm{i}\cdot\operatorname{Res}(f,1+\sqrt{2}\mathrm{i}) \;=\; \frac{\pi}{\sqrt{2}}.$$

For g, the singularities $z_0^g = 0$ and $z_1^g = -\mathrm{i}$ lie in the interior of $B_2(\mathrm{i}/2)$. With that:

$$\oint_\Gamma g(z)\;\mathrm{d}z \;=\; 2\pi\mathrm{i}\cdot(\operatorname{Res}(g,0)+\operatorname{Res}(g,-\mathrm{i})) \;=\; 2\pi\mathrm{i}\cdot\frac{\mathrm{i}}{4} \;=\; -\frac{\pi}{2}.$$

Problem 2. Compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} \, \mathrm{d}x.$$

Solution. The complex function

$$f(z) = \frac{1}{(z^2+1)(z^2+4)}$$

has singularities at $z_{1,2} = \pm \mathrm{i}$, $z_{3,4} = \pm 2\mathrm{i}$. All of them are simple poles. We compute the residues of the singularities with positive imaginary part:

$$\mathrm{Res}(f,\mathrm{i}) = \left.\frac{1}{(z+\mathrm{i})(z^2+4)}\right|_{z=\mathrm{i}} = \frac{1}{6\mathrm{i}}, \qquad \mathrm{Res}(f,2\mathrm{i}) = \left.\frac{1}{(z^2+1)(z+2\mathrm{i})}\right|_{z=2\mathrm{i}} = -\frac{1}{12\mathrm{i}}.$$

With that:

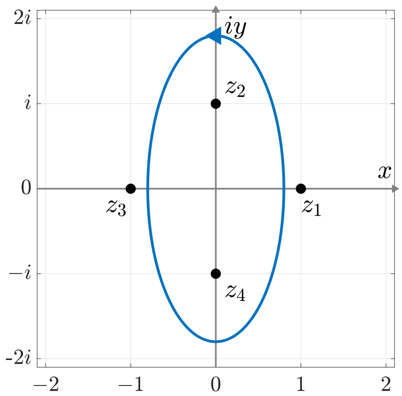
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} \, \mathrm{d}x = 2\pi\mathrm{i}\,(\mathrm{Res}(f,\mathrm{i}) + \mathrm{Res}(f,2\mathrm{i})) = \frac{\pi}{6}.$$

Problem 3. Compute

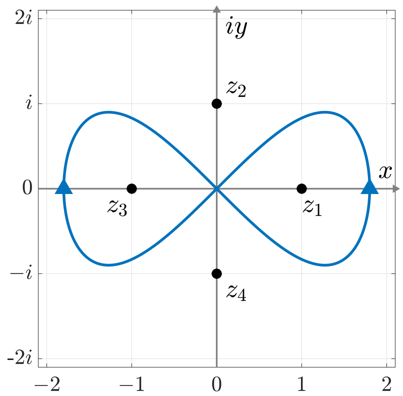
$$\int_\Gamma \frac{z^2}{(z^2+1)(z^2-1)}\,\mathrm{d}z$$

for the depicted curves Γ. The arrows show the direction of the curve and we assume that each curve is passed through once.

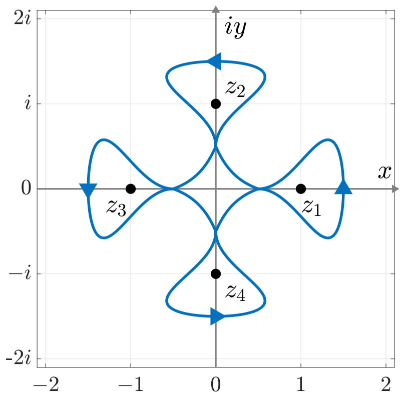
(a)



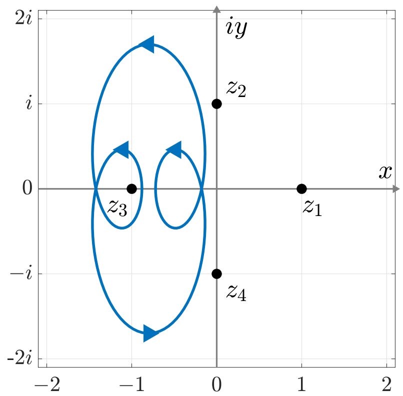
(b)



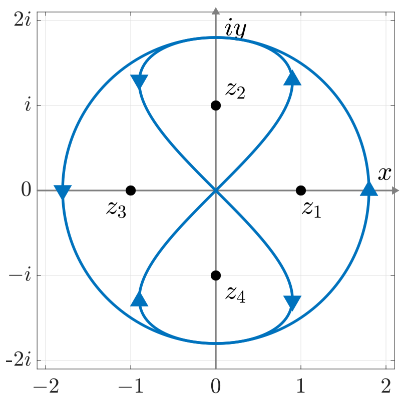
(c)



(d)



(e)



Solution. The singularities

$$f(z) = \frac{z^2}{(z^2+1)(z^2-1)}$$

lie at

$$z_1 = 1, \qquad z_2 = \mathrm{i}, \qquad z_3 = -1, \qquad z_4 = -\mathrm{i},$$

and are all simple poles. The corresponding residues are

$$\mathrm{Res}(f,1) = \tfrac{z^2}{(z^2+1)(z+1)}\Big|_{z=1} = \frac{1}{4}, \qquad \mathrm{Res}(f,\mathrm{i}) = \tfrac{z^2}{(z+\mathrm{i})(z^2-1)}\Big|_{z=\mathrm{i}} = \frac{1}{4\mathrm{i}},$$

$$\mathrm{Res}(f,-1) = \tfrac{z^2}{(z^2+1)(z-1)}\Big|_{z=-1} = -\frac{1}{4}, \qquad \mathrm{Res}(f,-\mathrm{i}) = \tfrac{z^2}{(z-\mathrm{i})(z^2-1)}\Big|_{z=-\mathrm{i}} = -\frac{1}{4\mathrm{i}}.$$

(a) The curve goes around z_2 and z_4 once in positive direction.

$$\int_\Gamma \frac{z^2}{(z^2+1)(z^2-1)}\,\mathrm{d}z = 2\pi\mathrm{i}\,(\mathrm{Res}(f,\mathrm{i}) + \mathrm{Res}(f,-\mathrm{i})) = 0.$$

(b) The curve goes around z_1 once in positive direction and around z_3 twice in negative direction.

$$\int_\Gamma \frac{z^2}{(z^2+1)(z^2-1)}\,\mathrm{d}z = 2\pi\mathrm{i}\,(\mathrm{Res}(f,1) - \mathrm{Res}(f,-1)) = \pi\mathrm{i}.$$

(c) The curve goes around all singularities once in positive direction.

$$\int_\Gamma \frac{z^2}{(z^2+1)(z^2-1)}\,\mathrm{d}z = 2\pi\mathrm{i}\,(\mathrm{Res}(f,1) + \mathrm{Res}(f,-1) + \mathrm{Res}(f,\mathrm{i}) + \mathrm{Res}(f,-\mathrm{i})) = 0.$$

(d) The curve goes around z_3 twice in positive direction.

$$\int_\Gamma \frac{z^2}{(z^2+1)(z^2-1)}\,\mathrm{d}z = 2\pi\mathrm{i}\,(2\mathrm{Res}(f,-1)) = -\pi\mathrm{i}.$$

(e) The curve goes around z_1 and z_3 once in positive direction, around z_2 twice in positive direction, and around z_4 once in positiver and once in negative direction.

$$\int_\Gamma \frac{z^2}{(z^2+1)(z^2-1)}\,\mathrm{d}z = 2\pi\mathrm{i}\,(\mathrm{Res}(f,1) + \mathrm{Res}(f,-1) + 2\mathrm{Res}(f,\mathrm{i}) + 0\cdot\mathrm{Res}(f,-\mathrm{i})) = \pi.$$