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Complex Functions for Students of Engineering Sciences

Worksheet 6 - Solutions

Problem 1. (*problem from an old exam, 7 points*) Consider the function

$$f(z) \;=\; \frac{1}{(z-2)^2(z+3)}$$

(a) How many Laurent series are there for f around the point $z_0 = 2$? In which rings do these series converge?

(b) For each of the rings from part (a), compute the Laurent series expansion of f that converges to f in the respective ring.

Solution.

(a) There are two Laurent series for f around $z_0 = 2$.

One converges inside the ring $R_1:\ 0 < |z-2| < 5$, and one converges inside the ring $R_2:\ 5 < |z-2|$.

(b) We write

$$f(z) = (z-2)^{-2} \cdot \frac{1}{z+3}$$

and expand $g(z) := \dfrac{1}{z+3} = \dfrac{1}{(z-2)+5}$ in each of the rings.

In R_1 we get:

$$g(z) \;=\; \frac{1}{(z-2)+5} \;=\; \frac{1}{5} \cdot \frac{1}{1-\left(-\frac{z-2}{5}\right)} \;=\; \frac{1}{5} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k (z-2)^k}{5^k}.$$

With that:

$$f(z) \;=\; (z-2)^{-2} \sum_{k=0}^{\infty} \frac{(-1)^k (z-2)^k}{5^{k+1}} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k (z-2)^{k-2}}{5^{k+1}} \;=\; \sum_{k=-2}^{\infty} \frac{(-1)^k}{5^{k+3}} (z-2)^k.$$

In R_2:

$$g(z) \;=\; \frac{1}{(z-2)+5} \;=\; \frac{1}{z-2} \cdot \frac{1}{1-\left(-\frac{5}{z-2}\right)} \;=\; \frac{1}{z-2} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k 5^k}{(z-2)^k}.$$

With that:

$$f(z) \;=\; (z-2)^{-2} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k 5^k}{(z-2)^{k+1}} \;=\; \sum_{k=0}^{\infty} (-5)^k (z-2)^{-k-3} \;=\; \sum_{k=-\infty}^{-3} (-5)^{-k-3} (z-2)^k.$$

Problem 2. Find the Laurent series expansions for the following functions. In particular, what is the coefficient c_{-1} in the expansion?

(a) $f(z) = \dfrac{\exp(z-2)}{z-2}$ around $z_0 = 2\,,$

(b) $f(z) = (z+1)^3 \cosh\left(\dfrac{1}{z+1}\right)$ around $z_0 = -1\,.$

Solution:

(a) For $|z-2| > 0$:

$$
\begin{aligned}
f(z) &= \frac{\exp(z-2)}{z-2} = \frac{1}{z-2}\left(1 + \frac{z-2}{1!} + \frac{(z-2)^2}{2!} + \frac{(z-2)^3}{3!} + \cdots\right) \\
&= \frac{1}{z-2} + 1 + \frac{z-2}{2!} + \frac{(z-2)^2}{3!} + \frac{(z-2)^3}{4!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(z-2)^{n-1}}{n!} = \sum_{n=-1}^{\infty} \frac{(z-2)^n}{(n+1)!} \qquad \Rightarrow \qquad c_{-1} = 1.
\end{aligned}
$$

(b) For $|z+1| > 0$:

$$
\begin{aligned}
f(z) &= (z+1)^3 \cosh\left(\frac{1}{z+1}\right) \\
&= (z+1)^3 \left(1 + \frac{1}{2!(z+1)^2} + \frac{1}{4!(z+1)^4} + \frac{1}{6!(z+1)^6} + \cdots\right) \\
&= (z+1)^3 + \frac{(z+1)}{2} + \frac{1}{4!(z+1)} + \frac{1}{6!(z+1)^3} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(z+1)^{-2n+3} = \sum_{n=-\infty}^{3} c_n (z+1)^n,
\end{aligned}
$$

where we have for $k = -\infty, \ldots, 1$:

$$
c_{2k} = 0, \qquad c_{2k+1} = \frac{1}{(3-(2k+1))!} \qquad \Rightarrow \qquad c_{-1} = \frac{1}{4!} = \frac{1}{24}.
$$

Problem 3. Which of the following statements are true, which are false? In each case, give a (short) explanation for your answer.

(a) Let f, g be analytic on $\mathbb{C}$. If there is a $z_0 \in \mathbb{C}$ with $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}_0$, the it holds that $f(z) = g(z)$ for all $z \in \mathbb{C}$.

(b) Let f be analytic on $\mathbb{C}$. If there is a $z_0 \in \mathbb{C}$ and a $m \in \mathbb{N}$, such that $f^{(n)}(z_0) = 0$ for all $n > m$, then f is a polynomial of degree at most m.

(c) Let f be analytic on an open disk around z_0 and let z_* be a point on the boundary of that disk, such that the Taylor series of f around z_0 diverges in z_*. Then f has a singularity at z_*.

Solution.

(a) The statement is *true*. From $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}_0$ it follow that the Taylor series of f and g around z_0 are the same. Since f and g are analytic on $\mathbb{C}$, both functions are represented on $\mathbb{C}$ by that series, and therefore are the same function on $\mathbb{C}$.

(b) The statement is *true*. With $f^{(n)}(z_0) = 0$ for all $n > m$, the Taylor series of f around z_0 is

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \sum_{n=0}^{m} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \\ &= f(z_0) + f'(z_0) \cdot (z - z_0) + \cdots + \frac{f^{(m)}}{m!}(z - z_0)^m. \end{aligned}$$

(c) The statement is *false*. A counter example: Consider $f(z) = \dfrac{1}{1 + z^2}$, $z_0 = 0$: The corresponding Taylor series

$$f(z) = \sum_{n=0}^{\infty} (-z^2)^n$$

converges for $|z| < 1$. The singularities $\pm \mathrm{i}$ lie on the boundary $|z| = 1$, so the radius of convergence cannot be larger than one. But also in $z_* = 1$ the series diverges,

$$\sum_{n=0}^{\infty} (-1^2)^n = \sum_{n=0}^{\infty} (-1)^n,$$

but f has no singularity at $z_* = 1$.