Fachbereich Mathematik der Universität Hamburg **SoSe 2025**
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Complex functions for students of engineering sciences

Worksheet 2 - Solutions

Problem 1. For each of the following sets, determine its image under the respective map. Draw a sketch of the image, or describe it with words.

(a)

$$D \;=\; \{z \in \mathbb{C} \;\mid\; 1 \leq |z+\mathrm{i}| \leq 2, \mathrm{Re}(z) \geq 0,\; \mathrm{Im}(z) \leq -1\},$$

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \frac{(z+\mathrm{i})^2}{1+\mathrm{i}}.$$

(b) (*problem from an old exam, 5 points*)

$$R \;=\; \left\{z = x + \mathrm{i}y \in \mathbb{C} \;\mid\; |x| \leq \frac{\ln(2)}{\pi},\; |y| \leq \frac{1}{2}\right\},$$

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \cdot \mathrm{e}^{\pi z}.$$

(c)

$$V \;=\; \{z = x + \mathrm{i}y \in \mathbb{C} \;\mid\; 0 < x \leq 1,\; 0 < y \leq 1,\; x^2 + y^2 \leq 1\},$$

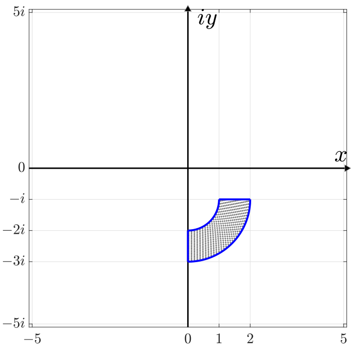
$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad f(z) = \frac{1}{z}.$$

Solution.

(a)

The set D is the lower right quarter of a ring with center $z_0 = -\mathrm{i}$, inner radius 1 and outer radius 2.

$$D = \left\{z \in \mathbb{C} \;\mid\; z = -\mathrm{i} + r\mathrm{e}^{\mathrm{i}\varphi},\; 1 \leq r \leq 2,\; \varphi \left[-\frac{\pi}{2}, 0\right]\right\}$$



D

We decompose f into

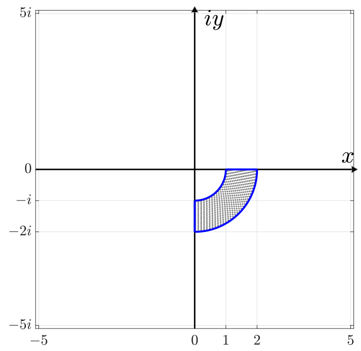
$$f = f_3 \circ f_2 \circ f_1,$$

with

$$u = f_1(z) = z+i, \quad v = f_2(u) = u^2, \quad w = f_3(v) = \frac{v}{1+i}.$$

Then f_1 is a translation in direction i and thus maps the center of the quarter ring to 0.

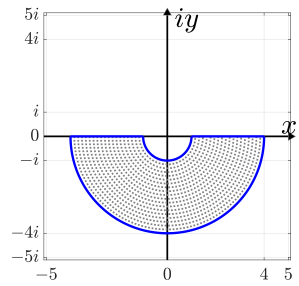
$$f_1(D) = \left\{u \in \mathbb{C} \ \mid \ u = r\mathrm{e}^{\mathrm{i}\varphi},\ 1 \le r \le 2,\ \varphi \left[-\frac{\pi}{2}, 0\right]\right\}$$



$f_1(D)$

Next, we apply f_2. Squaring of a complex number means squaring the radius and doubling the angle. We get half of a ring with inner radius 1 and outer radius 4, with angles between $-\pi$ and 0.

$$f_2(f_1(D)) = \left\{v \in \mathbb{C} \ \mid \ v = s\mathrm{e}^{\mathrm{i}\psi},\ 1 \le s \le 4,\ \psi\, [-\pi, 0]\right\}$$



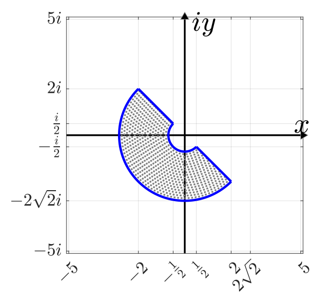
$f_2(f_1(D))$

Finally, we apply f_3. Writing

$$1 + \mathrm{i} = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}, \qquad \text{or} \qquad \frac{1}{1+\mathrm{i}} = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}},$$

we see that we have to multiply the radii with $1/\sqrt{2}$ and rotate by $-\pi/4$. The result is a half-ring with inner radius $1/\sqrt{2}$, outer radius $2\sqrt{2}$, and angles between $-5\pi/4$ and $-\pi/4$.

$$f(D) =$$
$$\left\{w \in \mathbb{C} \ \mid \ w = t\mathrm{e}^{\mathrm{i}\theta},\ \frac{1}{\sqrt{2}} \le t \le 2\sqrt{2},\ \theta \in \left[-\frac{5\pi}{4}, -\frac{\pi}{4}\right]\right\}$$



$f(D) = f_3(f_2(f_1(D)))$

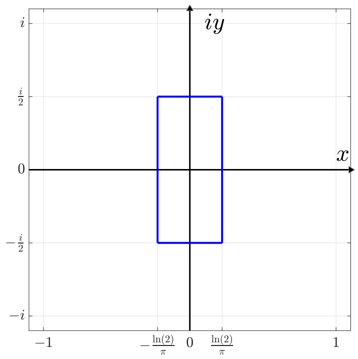
(b)

The set R is a rectangle. We decompose f into

$$f = f_2 \circ f_1,$$

with

$$u = f_1(z) = \mathrm{e}^{\pi z}, \quad w = f_2(u) = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \cdot u.$$



R

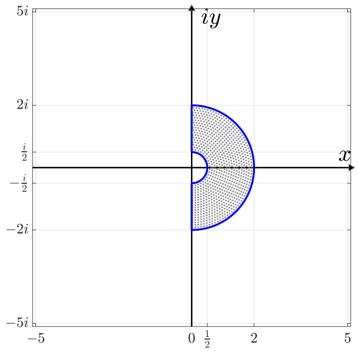
For $z = x + \mathrm{i}y \in R$ it holds

$$u \;=\; f_1(z) \;=\; \mathrm{e}^{\pi(x+\mathrm{i}y)} \;=\; \mathrm{e}^{\pi x} \cdot \mathrm{e}^{\mathrm{i}\pi y}.$$

With $x \in [-\frac{\ln(2)}{\pi}, \frac{\ln(2)}{\pi}]$ and $y \in [-\frac{1}{2}, \frac{1}{2}]$, we get $u = s\mathrm{e}^{\mathrm{i}\theta}$ with $s = \mathrm{e}^{\pi x}$ and $\theta = \pi y$, where

$$s \in [\mathrm{e}^{-\ln(2)}, \mathrm{e}^{\ln(2)}] \;=\; \left[\frac{1}{2}, 2\right], \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

We get a half-ring with inner radius $1/2$, outer radius 2 and angles between $-\pi/2$ and $\pi/2$.



$f_1(R)$

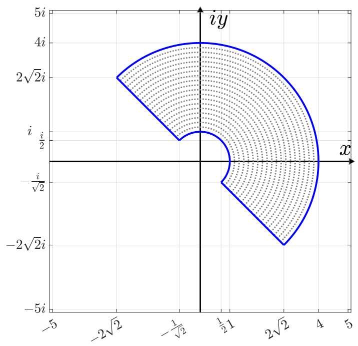
Finally, $w = f_2(u)$ leads to a stretching by a factor of 2 and a rotation by $\pi/4$. For $u \in f_1(R)$ we arrive at

$$w \;=\; f_2(u) \;=\; 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \cdot s\mathrm{e}^{\theta} \;=\; 2s\mathrm{e}^{\mathrm{i}(\theta+\frac{\pi}{4})} \;=\; r\mathrm{e}^{\mathrm{i}\varphi}$$

with

$$r = 2s \in [1, 4], \qquad \varphi = \theta + \frac{\pi}{4} \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right].$$

Thus, we have a half-ring with inner radius 1, outer radius 4 and angles between $-\pi/4$ and $3\pi/4$.

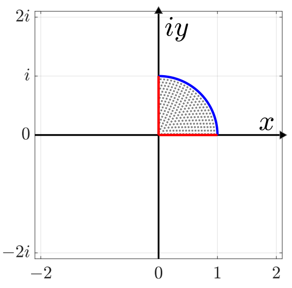


$f(R) = f_2(f_1(R))$

(c)

The set V is the upper right quarter of the closed unit disk, without zero. We write

$$V = \left\{ z = re^{i\varphi} \;\mid\; r \in (0,1],\; \varphi \in \left(0, \frac{\pi}{2}\right) \right\}$$



V

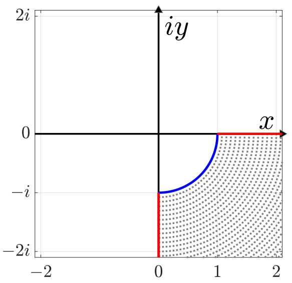
For $z = re^{i\varphi}$ we get

$$f(z) = \frac{1}{z} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}\varphi} = s\mathrm{e}^{i\theta}.$$

with

$$s = \frac{1}{r} \in [1, \infty), \qquad \theta = -\varphi \in \left(-\frac{\pi}{2}, 0\right).$$

The result is the lower right quadrant, without the open unit disk.



$f(V)$

Problem 2: Determine all complex solutions $z \in \mathbb{C}$ of the following equations.

(a) (*problem from an old exam, 3 points*) $\quad 2\mathrm{e}^{3z} - \dfrac{\sqrt{2}(1+\mathrm{i})}{\mathrm{e}^z} = 0,$

(b) $\quad z^4 = 8(1 + \mathrm{i}\sqrt{3})\,,$

Solutions:

(a) It holds

$$2\mathrm{e}^{3z} - \frac{\sqrt{2}(1+\mathrm{i})}{\mathrm{e}^z} = 0 \quad \Leftrightarrow \quad \mathrm{e}^{4z} = \frac{\sqrt{2}}{2}(1+\mathrm{i}).$$

For $\;w := \mathrm{e}^{4z}\;$ we require

$$w \;=\; \mathrm{e}^{4z} \;=\; \mathrm{e}^{4(x+\mathrm{i}y)} \;=\; \mathrm{e}^{4x} \cdot \mathrm{e}^{\mathrm{i}4y} \;\stackrel{!}{=}\; \frac{\sqrt{2}}{2}(1+\mathrm{i}).$$

Thus,

$$|w| \;=\; |e^{4x}\cdot e^{i4y}| \;=\; e^{4x} \;=\; \left|\frac{\sqrt{2}}{2}(1+i)\right| \;=\; \left|\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right| \;=\; \sqrt{\frac{1}{2}+\frac{1}{2}}=1$$
$$\Leftrightarrow \quad 4x \;=\; \ln(1) \quad \Leftrightarrow \quad x \;=\; 0.$$

It remains

$$e^{i4y} \;=\; \left|\frac{\sqrt{2}}{2}(1+i)\right| \quad \Leftrightarrow \quad \arg\left(e^{i4y}\right) \;=\; \arg\left(\frac{\sqrt{2}}{2}(1+i)\right)+2k\pi$$
$$\Leftrightarrow \quad 4y \;=\; \frac{\pi}{4}+2k\pi$$
$$\Leftrightarrow \quad y \;=\; \frac{\pi}{16}+\frac{k}{2}\pi, \quad k\in\mathbb{Z}.$$

(b) We have

$$z^4 \;=\; 8(1+i\sqrt{3}) \quad \Leftrightarrow \quad |z^4| \;=\; |z|^4 \;=\; \sqrt{8^2+3\cdot 8^2} \;=\; \sqrt{4\cdot 8^2} \;=\; 16$$
$$\Leftrightarrow \quad |z| \;=\; 2.$$

In polar coordinates, $z = re^{i\varphi}$, it follows

$$z^4 \;=\; r^4e^{i4\varphi} \;=\; 16e^{i4\varphi} \;=\; 16\left(\frac{1}{2}+i\frac{\sqrt{3}}{3}\right) \;=\; 16e^{i\frac{\pi}{3}}.$$

Therefore,

$$4\varphi \;=\; \frac{\pi}{3}+2k\pi \quad \Leftrightarrow \quad \varphi \;=\; \frac{\pi}{12}+\frac{k}{2}\pi.$$

We get four solutions, each with radius 2, and with angles in $\{\frac{1}{12}\pi, \frac{7}{12}\pi, \frac{13}{12}\pi, \frac{19}{12}\pi\}$.