Fachbereich Mathematik der Universität Hamburg **SoSe 2025**
Prof. Dr. A. Iske, Dr. C. Goetz

Complex Functions for students of engineering sciences

Worksheet 1 - Solutions

Problem 1: Consider the complex numbers

$$z_1 = 2 + 3\mathrm{i}, \qquad z_2 = -4 + 5\mathrm{i}.$$

Compute the following in *Cartesian coordinates*:

(a) $z_1 + z_2\,, \quad |z_1 + z_2|\,, \quad 2z_1 - 3\mathrm{i}z_2\,, \quad 2\bar{z}_1 - 3\mathrm{i}\bar{z}_2\,,$

(b) $z_1 \cdot z_2\,, \quad \bar{z}_1 \cdot \bar{z}_2\,, \quad z_1^2 \cdot z_2^2\,, \quad \mathrm{Re}(z_1^2) \cdot \mathrm{Im}(z_2^2)\,,$

(c) $\dfrac{z_1}{z_2}\,, \quad \dfrac{\mathrm{Im}(z_1)}{\mathrm{Re}(z_2)}\,.$

Solution: We compute directly:

(a) $z_1 + z_2 = 2 + 3\mathrm{i} + (-4 + 5\mathrm{i}) = -2 + 8\mathrm{i}\,,$

$|z_1 + z_2| = |-2 + 8\mathrm{i}| = \sqrt{2^2 + 8^2} = \sqrt{68}\,,$

$2z_1 - 3\mathrm{i}z_2 = 2(2 + 3\mathrm{i}) - 3\mathrm{i}(-4 + 5\mathrm{i}) = 19 + 18\mathrm{i}\,,$

$2\bar{z}_1 - 3\mathrm{i}\bar{z}_2 = 2(2 - 3\mathrm{i}) - 3\mathrm{i}(-4 - 5\mathrm{i}) = -11 + 6\mathrm{i}$

(b) $z_1 \cdot z_2 = (2 + 3\mathrm{i})(-4 + 5\mathrm{i}) = -23 - 2\mathrm{i}\,,$

$\bar{z}_1 \cdot \bar{z}_2 = \overline{z_1 \cdot z_2} = -23 + 2\mathrm{i}\,,$

$z_1^2 \cdot z_2^2 = (2 + 3\mathrm{i})^2(-4 + 5\mathrm{i})^2 = (-5 + 12\mathrm{i})(-9 - 40\mathrm{i}) = 525 + 92\mathrm{i}\,,$

$\mathrm{Re}(z_1^2) \cdot \mathrm{Im}(z_2^2) = -5 \cdot (-40) = 200\,,$

(c) $\dfrac{z_1}{z_2} = \dfrac{2 + 3\mathrm{i}}{-4 + 5\mathrm{i}} = \dfrac{(2 + 3\mathrm{i})(-4 - 5\mathrm{i})}{(-4 + 5\mathrm{i})(-4 - 5\mathrm{i})} = \dfrac{7}{41} - \dfrac{22}{41}\mathrm{i}$

$\dfrac{\mathrm{Im}(z_1)}{\mathrm{Re}(z_2)} = -\dfrac{3}{4}$

Problem 2: Consider the complex numbers

$$z_1 = 1 + \mathrm{i}\sqrt{3}, \qquad z_2 = 1 - \mathrm{i}\sqrt{3}, \qquad z_3 = -1 - \mathrm{i}\sqrt{3}, \qquad z_4 = -1 + \mathrm{i}\sqrt{3}.$$

(a) Sketch the position of these points in the complex plane and compute their *polar coordinates* ($z = re^{i\varphi}$).

(b) Compute $z_1 \cdot z_4$ and z_1^7.

(c) Compute $\dfrac{z_1^2 \cdot z_2}{z_3 \cdot \bar{z}_4}$.

Solution:

(a) In polar coordinates, $z_k = r_k \mathrm{e}^{\mathrm{i}\varphi_k}$, the radii satisfy

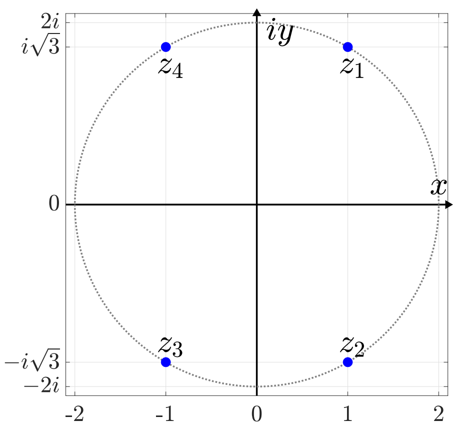
$$r_k = \sqrt{1^2 + (\sqrt{3})^2} = 2, \qquad k = 1, 2, 3, 4.$$

We compute the angles as

$$\begin{array}{llcll}
x_1 = 1 > 0, & y_1 = \sqrt{3} > 0 & \Rightarrow & \varphi_1 = \arctan(\sqrt{3}) & = \frac{\pi}{3}, \\
x_2 = 1 > 0, & y_2 = -\sqrt{3} < 0 & \Rightarrow & \varphi_2 = \arctan(-\sqrt{3}) & = -\frac{\pi}{3}, \\
x_3 = -1 < 0, & y_3 = -\sqrt{3} < 0 & \Rightarrow & \varphi_3 = \arctan(\sqrt{3}) - \pi & = -\frac{2\pi}{3}, \\
x_4 = -1 < 0, & y_4 = \sqrt{3} > 0 & \Rightarrow & \varphi_4 = \arctan(-\sqrt{3}) + \pi & = \frac{2\pi}{3}.
\end{array}$$

So we get

$$z_1 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}, \qquad z_2 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}},$$
$$z_3 = 2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}, \qquad z_4 = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}.$$



(b) Using the rules for multiplication in polar coordinates we find

$$z_1 \cdot z_4 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} \cdot 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} = (2 \cdot 2) \cdot \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)} = 4 \cdot \mathrm{e}^{\mathrm{i}\pi} = -4,$$

and

$$z_1^7 = \left(2\mathrm{e}^{i\frac{\pi}{3}}\right)^7 = 2^7 \cdot \mathrm{e}^{i\frac{\pi}{3}\cdot 7} = 128 \cdot \mathrm{e}^{\mathrm{i}\left(2\pi + \frac{\pi}{3}\right)} = 128\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}.$$

(c) We compute nominator and denominator separately:

$$z_1^2 \cdot z_2 \;=\; \left(2e^{i\frac{\pi}{3}}\right)^2 \cdot 2e^{-i\frac{\pi}{3}} \;=\; 8e^{i\left(\frac{2\pi}{3}-\frac{\pi}{3}\right)} \;=\; 8e^{i\frac{\pi}{3}}$$

and

$$z_3 \cdot \bar{z}_4 \;=\; 2e^{-i\frac{2\pi}{3}} \cdot 2e^{-i\frac{2\pi}{3}} \;=\; 4e^{-i\frac{4\pi}{3}} \;=\; 4e^{i\frac{2\pi}{3}}.$$

With that:

$$\frac{z_1^2 \cdot z_2}{z_3 \cdot \bar{z}_4} \;=\; \frac{8}{4} \cdot e^{i\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)} \;=\; 2e^{-i\frac{\pi}{3}} \;=\; z_2.$$

Problem 3: Sketch the following sets in the complex plane or describe them with words.

(a) $M_1 = \{z \in \mathbb{C} \;\mid\; |z - 1 + 3i| \leq 2\}$,

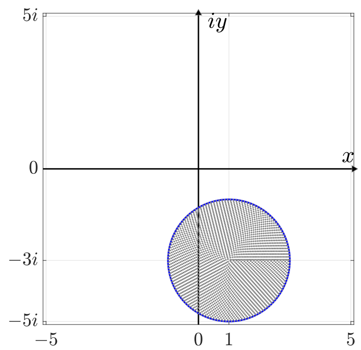
(b) $M_2 = \{z \in \mathbb{C} \;\mid\; |z + i| = |z - 3i|\}$,

(c) $M_3 = \{0\} \cup \{z \in \mathbb{C} \setminus \{0\} \;\mid\; \mathrm{Re}\left(\frac{z}{\bar{z}}\right) = 0\}$,

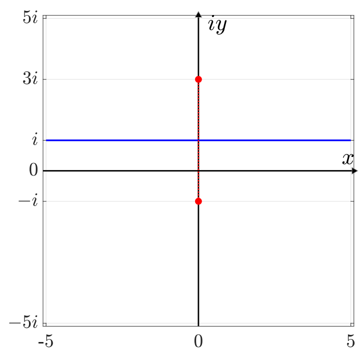
(d) $M_4 = \{z = re^{i\varphi} \in \mathbb{C} \;\mid\; r \in (2,5),\; \varphi \in (-\pi/6, \pi/6)\}$.

Solution:

(a) M_1 is a disk with center $z_0 = 1 - 3i$ and radius 2 (including the boundary).



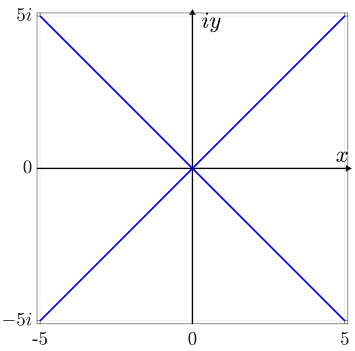
(b) M_2 is the set of all points with the same distance to both $z_1 = -i$ and to $z_2 = 3i$. Therefore, it is a straight line that is orthogonal to the line connecting z_1 and z_2 and passes through the midpoint of that line.
The line connecting z_1 and z_2 runs along the imaginary axis and its midpoint is i, i.e. M_2 is the line parallel to the real axis that passes through i,
$M_2 = \{z \in \mathbb{C} \;\mid\; \mathrm{Im}(z) = 1\}$.



(c) With $z = re^{i\varphi}$, $r \neq 0$ we have $\bar{z} = re^{-\mathrm{i}\varphi}$ and $\frac{z}{\bar{z}} = \mathrm{e}^{2\mathrm{i}\varphi}$. Thus, $\mathrm{Re}(z/\bar{z}) = \cos(2\varphi)$. Then we get

$$\mathrm{Re}\left(\frac{z}{\bar{z}}\right) = \cos(2\varphi) = 0 \quad \Rightarrow \quad 2\varphi = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$
$$\Rightarrow \quad \varphi = \frac{\pi}{4} + \frac{k\pi}{,} \quad k \in \mathbb{Z}.$$

M_3 consists of the lines $\mathrm{Re}(z) = \mathrm{Im}(z)$ and $\mathrm{Re}(z) = -\mathrm{Im}(z)$.



(d) M_4 is a segment of the ring with inner radius 2 and outer radius 5, with angles between $-\pi/6$ and $\pi/6$ (without the boundary).