SoSe 2025

Complex Functions
for students of engineering sciences

Auditorium Exercise 1:

Representing complex numbers, elementary calculations, geometry of the complex plane

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Administrative notes

Contact:

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- Office hours: Tue., 12:00 - 13:00, E - 4.013 (in weeks with CoFu - exercises)

Schedule:

- Lecture: Fri., 09:45 - 11:15 (D2.022), weekly, start: 11.04.
- Auditorium exercises: Fri., 11:30 - 13:00 (D2.022), every other week, alternating with Differential Equations 2, start: 11.04.
- Group exercises: every other week, alternating with Differential Equations 2, start: 14.04.

Administrative notes

Materials:

- Lehrexport UHH:
  `https://www.math.uni-hamburg.de/teaching/export/tuhh/cm/kf/25/lm.html.en`
- In Stud.IP under: Komplexe Funktionen (EN) (HÜ)

Procedure:

- Every other week there will be a worksheet with problems that we solve together during the groups exercises.
- You should take a look at the problems before class and at the solutions afterwards. If you only think about the problems during class, that is often not enough.
- In the auditorium exercise we repeat topics from the lecture that you need to solve the exercises.
- Every other week there will be sheet with home work. You can submit your solutions be e-mail (detail in the group exercises).

1. Why do we need complex numbers and functions?

Complex numbers let us solve equations
that have no solution in the field of real numbers.

$$z+1=0 \quad \rightarrow \quad \text{no solution in } \mathbb{N}, \text{ but in } \mathbb{Z}$$
$$2z-1=0 \quad \rightarrow \quad \text{no solution in } \mathbb{Z}, \text{ but in } \mathbb{Q}$$
$$z^2-2=0 \quad \rightarrow \quad \text{no solution in } \mathbb{Q}, \text{ but in } \mathbb{R}$$

$$z^2+1=0 \quad \rightarrow \quad \text{no solution in } \mathbb{R}, \text{ but in } \mathbb{C}$$

→ **Complex numbers are nothing mysterious,
they simply help us to solve more equations!**

Complex numbers can be useful for solving **real** problems.

<u>**Example:**</u> DGL 1, HA 4: $$u'(t) = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} u(t).$$

General complex solution:

$$w(t) = c_1 \mathrm{e}^{\mathrm{i}\frac{t}{3}} \begin{pmatrix} 1-\mathrm{i} \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{-\mathrm{i}\frac{t}{3}} \begin{pmatrix} 1+\mathrm{i} \\ 1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{C}$$

General real solution:

$$u(t) = d_1 \begin{pmatrix} \sin(t/3) + \cos(t/3) \\ \cos(t/3) \end{pmatrix} + d_2 \begin{pmatrix} \sin(t/3) - \cos(t/3) \\ \sin(t/3) \end{pmatrix}, \quad d_1, d_2 \in \mathbb{R}$$

$\rightarrow$ **It is much easier to find the complex solution!**

Complex analysis helps us to understand real functions better.

This will be an easy exercise by the end of the semester!

Why is $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \pi\ ?$$

Why does the Taylor expansion of $f(x) = \dfrac{1}{1+x^2}$ aroand zero,

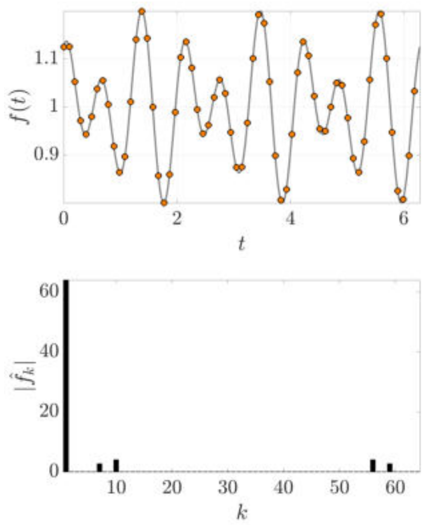
$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots,$$

$f(z) = \dfrac{1}{1+z^2}$, $z \in \mathbb{C}$ has a singularity at $z = i$, $|i| = 1$

only converge for $|x| < 1$?

→ **With a little complex analysis these questions can be answered more easily than with purely real tools!**

The real function "inherits the problems" of the complex function.



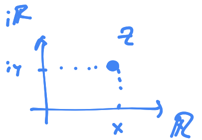
discrete Fourier transform (DFT)

Complex numbers are an extremely important tool in signal processing!

2. Representing complex numbers

Cartesian coordinates and polar coordinates

Just as in $\mathbb{R}^2$:



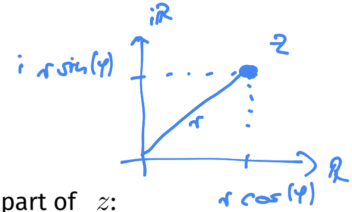
$(x,y) \in \mathbb{R}^2$

For $z \in \mathbb{C}$ we write in **Cartesian coordinates**: $z = x + \mathrm{i}y$

$x, y \in \mathbb{R}$, i : imaginary unit , $\mathrm{i}^2 = -1$.

Real part of z: $\mathrm{Re}(z) = x$, Imaginary part of z: $\mathrm{Im}(z) = y$,

Modulus of z: $|z| = \sqrt{x^2 + y^2}$.



In **polar coordinates**: $z = r(\cos(\varphi) + \mathrm{i}\sin(\varphi))$

$x = r\cos(\varphi), \quad y = r\sin(\varphi)$.

Real part of z: $\mathrm{Re}(z) = r\cos(\varphi)$, Imaginary part of z:
$\mathrm{Im}(z) = r\sin(\varphi)$,

Modulus z: $|z| = r$.

The angle in the polar coordinates is called the argument.

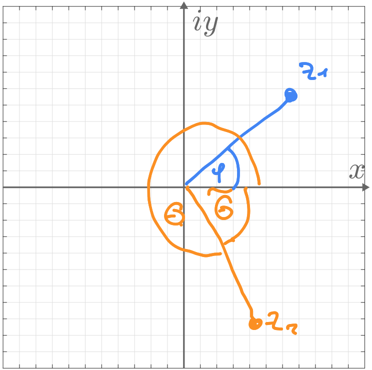
2π - periodic

For $z = r(\cos(\varphi) + \mathrm{i}\sin(\varphi))$ we call $\varphi = \arg(z)$ the argument of z.
Note: $\arg(z)$ is only determined up to multiples of 2π..

It is often convenient to allow negative angles.

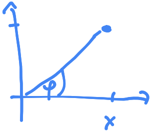
φ : angle between real axis and position vector (mathematically positive).

$\theta + |\tilde{\theta}| = 2\pi$



From Cartesian coordinates to polar coordinates

$\tan(\varphi) = \frac{y}{x}$



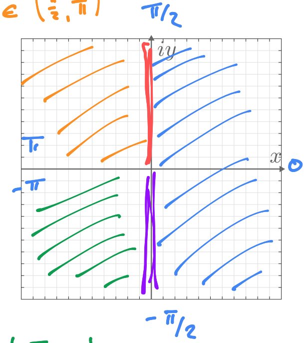
Typically, we choose $\arg(z) \in (-\pi, \pi)$ (principal value).

For given $z = x + iy \neq 0$ we can compute:

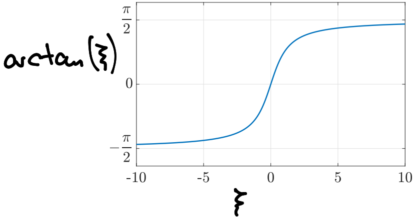
$$\varphi = \begin{cases} \arctan\left(\frac{y}{x}\right), & x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi, & x < 0,\ y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi, & x < 0,\ y < 0, \\ \frac{\pi}{2}, & x = 0,\ y > 0, \\ -\frac{\pi}{2}, & x = 0,\ y < 0, \end{cases}$$

$\varphi \in (\frac{\pi}{2}, \pi)$

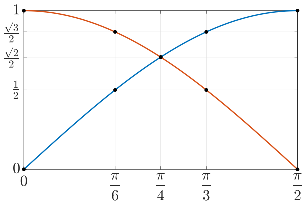
$\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$



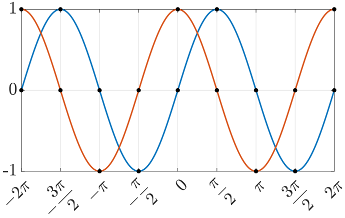
$\varphi \in (-\frac{\pi}{2}, \pi)$



A few nice-to-know values



sin and cos on $[0, \pi/2]$



sin and cos on $[-2\pi, 2\pi]$

$$\tan(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

radiant	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\underline{\frac{\pi}{3}}$	$\frac{\pi}{2}$
degree	$0°$	$30°$	$45°$	$60°$	$90°$
sin	$0 = \frac{\sqrt{0}}{2}$	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$1 = \frac{\sqrt{4}}{2}$
cos	$1 = \frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$0 = \frac{\sqrt{0}}{2}$

$$\arctan(\sqrt{3}) = \frac{\pi}{3}$$

$$\arctan(1) = \frac{\pi}{4}$$

$$\arctan(0) = 0$$

A very helpful representation comes from the **Euler Formula**.

Euler Formula

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi).$$

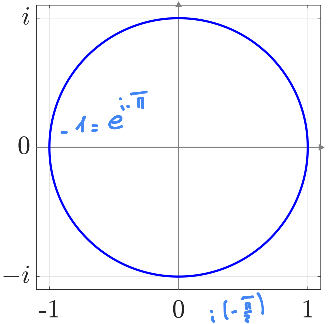
From the polar coordinates
we get for $z \in \mathbb{C}$:

$$z = r\left(\cos(\varphi) + i\sin(\varphi)\right)$$

$$\Rightarrow \quad z = re^{i\varphi}$$

$r \in (0, \infty), \quad \varphi \in (-\pi, \pi]$

$i = e^{i\cdot\frac{\pi}{2}}$



$-1 = e^{i\cdot\pi}$

$1 = e^{i\cdot 0}$

$-i = e^{i(-\frac{\pi}{2})}$

$-i = -1\cdot i$

$= e^{i\pi}\cdot e^{i\cdot\frac{\pi}{2}}$

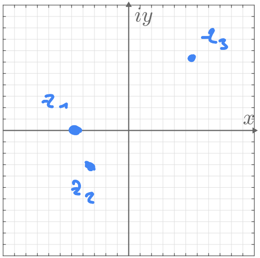
$= e^{i\frac{3\pi}{2}}$

$= e^{i(\frac{3\pi}{2} - 2\pi)}$

$= e^{i(-\frac{\pi}{2})}$

$= e$

Cartesian coordinates $\longrightarrow$ polar coordinates



$$z_1 = -4 + i \cdot 0$$

$$r = \left((-4)^2 + 0^2\right)^{1/2} = \sqrt{16} = 4$$

$$x < 0,\ y \geq 0 \Rightarrow \varphi = \arctan\left(\frac{y}{x}\right) + \pi = \arctan(0) + \pi = \pi$$

$$\Rightarrow z_1 = 4e^{i\pi}$$

$$z_2 = \sqrt{8}(-1 - \mathrm{i})$$

$$r = \left(\sqrt{8}^2 + \sqrt{8}^2\right)^{1/2} = \sqrt{16} = 4$$

$$x = -\sqrt{8} < 0,\ y = -\sqrt{8} < 0 \Rightarrow \varphi = \arctan(1) - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

$$\Rightarrow z_2 = 4e^{-i\frac{3\pi}{4}}$$

$$z_3 = 6 + 8\mathrm{i}$$

$$r = \left(6^2 + 8^2\right)^{1/2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

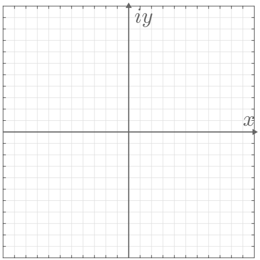
$$x > 0,\ y > 0 \Rightarrow \varphi = \arctan\left(\frac{4}{3}\right) \approx 0.927$$

Sometimes we get numbers that aren't easy to do calculations with.

polar coordinates $\longrightarrow$ Cartesian coordinates

$$z_4 = 3e^{i\frac{\pi}{3}} = 3\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$
$$= 3\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

$$z_5 = 2e^{-i\frac{\pi}{6}} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$
$$= 2\left(\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$$
$$= 2\left(\frac{\sqrt{3}}{2} - i\cdot\frac{1}{2}\right) = \sqrt{3} - i.$$



3. Elementary calculations

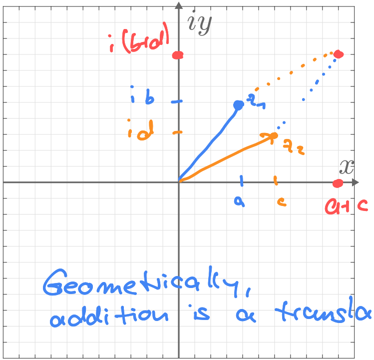
Addition of complex numbers is best done in **Cartesian coordinates**.

Addition works component-wise (just as in $\mathbb{R}^2$).

$$z_1 = a + \mathrm{i}b, \quad z_2 = c + \mathrm{i}d:$$

$$z_1 + z_2 = (a+c) + \mathrm{i}(b+d)$$

For adding $re^{i\varphi}$ and $se^{i\theta}$, we would first need to compute their Cartesian coordinates.



Geometrically, addition is a translation.

Multiplication / division of complex numbers is best done in **polar coordinates**.

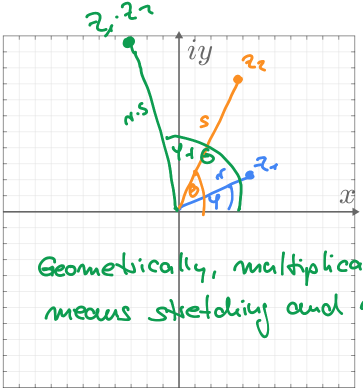
In polar coordinates: $z_1 = re^{i\varphi}, \quad z_2 = se^{i\theta} \neq 0$:

$$z_1 \cdot z_2 = \underline{r}e^{i\varphi} \cdot \underline{s}e^{i\theta} = \underline{rs} \cdot e^{i(\underline{\varphi+\theta})},$$

$$\frac{z_1}{z_2} = \frac{r}{s}e^{i(\varphi-\theta)}$$

Stretching / compression

rotation



Geometrically, multiplication means stretching and rotation.

In Cartesian coordinates:

$$z_1 = a + ib, \quad z_2 = c + id \neq 0:$$

$$z_1 \cdot z_2 = (ac - bd) + i(ad + bc)$$

$$\frac{z_1}{z_2} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right)$$

Of course you can do this, but it's not immediately obvious what is going on here.

Calculations in Cartesian coordinates

Example: $z_1 = 2 - 4\mathrm{i}, \qquad z_2 = -3 + 2\mathrm{i}$

$$z_1 \cdot z_2 = (2-4i)(-3+2i) = -6 + 4i + 12i - 8i^2 = 2 + 16i$$

$$\frac{z_1}{z_2} = \frac{2-4i}{-3+2i} \cdot \frac{-3-2i}{-3-2i} = \frac{1}{13}(2-4i)(-3-2i) = \frac{1}{13}\left(-6-4i+12i+8i^2\right)$$

$$= -\frac{14}{13} + i\,\frac{8}{13}$$

expand by the conjugate of the denominator

This isn't too hard, but more work than multiplication in polar coordinates.

Calculations in polar coordinates

Example: $z_3 = 3e^{i\frac{\pi}{4}}, \quad z_4 = 2e^{-i\frac{\pi}{8}}$

$$z_3 \cdot z_4 = 3 \cdot 2 \cdot e^{i\left(\frac{\pi}{4} - \frac{\pi}{8}\right)} = 6 e^{i\frac{\pi}{8}}$$

$$\frac{z_3}{z_4} = \frac{3}{2} e^{i\left(\frac{\pi}{4} - \left(-\frac{\pi}{8}\right)\right)} = \frac{3}{2} e^{i\frac{3\pi}{8}}$$

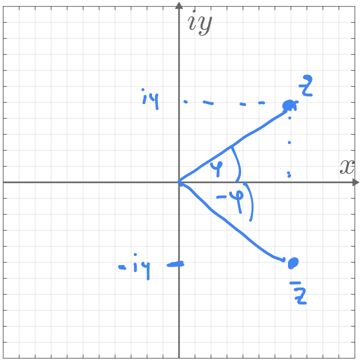
Die conjugate complex number

For $z = x + \mathrm{i}y$ we call $\bar{z} = x - \mathrm{i}y$
the **conjugate** complex number.

For $z = r\mathrm{e}^{\mathrm{i}\varphi}$: $\bar{z} = r\mathrm{e}^{\underline{-}\mathrm{i}\varphi}$.

We have

$$\begin{aligned} \mathrm{Re}(z) &= \frac{1}{2}(z + \bar{z}) \\ \mathrm{Im}(z) &= \frac{1}{2i}(z - \bar{z}) \\ |z| &= \sqrt{z \cdot \bar{z}} \end{aligned}$$



Geometrically, conjugation is a reflection on the real axis.

4. On the geometry of the complex plane

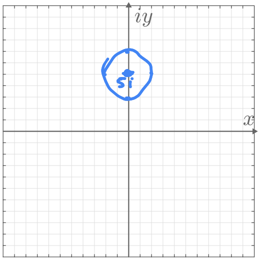
How do the sets described by the following formulas look like?

$$D_1 = \{z \in \mathbb{C} \mid |z - 5\mathrm{i}| = 2\}$$

Circle around 5i with radius 2

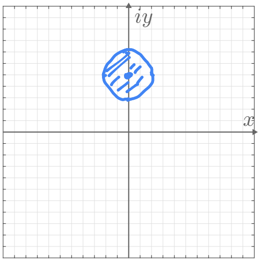
In general: $\{z \in \mathbb{C} \mid |z - z_0| = R\}$

center radius



$$D_2 = \{z \in \mathbb{C} \mid |z - 5\mathrm{i}| \leq 2\}$$

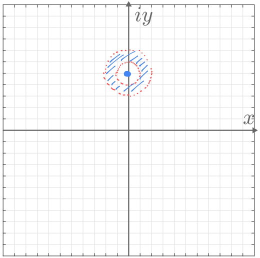
Disk around 5i with radius 2, including the boundary.



How do the sets described by the following formulas look like?

$$D_3 = \{z \in \mathbb{C} \mid 1 < |z - 5\mathrm{i}| < 2\}$$

Ring around $5i$ with inner radius 1, outer radius 2, without the boundary.



$|z-0|$

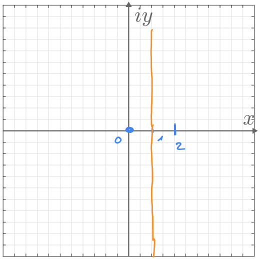
$$D_4 = \{z \in \mathbb{C} \mid |z - 2| = |z|\}$$

distance to 2

distance to 0

Line that is orthogonal to the line connecting 0 and 2 and passes through the midpoint of the connecting line.

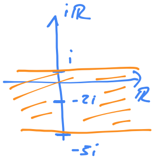
$D_4 = \{ z \in \mathbb{C} \mid \operatorname{Re}(z) = 1 \}$



Which formulas describe the following sets?

M_1: Strip parallel to the real axis, width 6, symmetric to $z_0 = -2\mathrm{i}$.

$M_1 = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) \in [-5, 1] \}$



M_2: Open ring around $z_0 = -2\mathrm{i}$ with inner radius 2, and outer radius 3.

$M_2 = \{ z \in \mathbb{C} \mid 2 < |z + 2i| < 3 \}$

$|z - z_0|$

Which formulas describe the following sets?

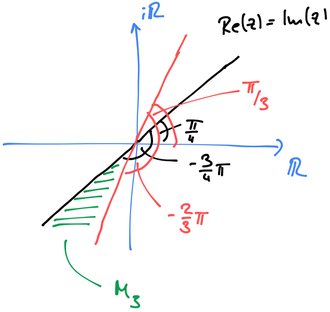
M_3: Sector between the lines with $\mathrm{Re}(z) = \mathrm{Im}(z)$ and $\sqrt{3} \cdot \mathrm{Re}(z) = \mathrm{Im}(z)$ in the lower half-plane.

$\mathrm{Re}(z) = \mathrm{Im}(z) \Rightarrow x = y \Rightarrow \varphi_1 = \arctan(1) - \pi$

$= \frac{\pi}{4} - \pi = -\frac{3}{4}\pi$

$\sqrt{3}\,\mathrm{Re}(z) = \mathrm{Im}(z) \Rightarrow \sqrt{3}\,x = y \Rightarrow \varphi_2 = \arctan(\sqrt{3}) - \pi$

$= \frac{\pi}{3} - \pi = -\frac{2}{3}\pi$



$M_3 = \{ z = re^{i\varphi} \in \mathbb{C} \mid r \geq 0,\ \varphi \in [-\frac{3}{4}\pi, -\frac{2}{3}\pi] \}$

M_4: Domain outside the closed disks with radius $R = \frac{3}{2}$ around the points 1 and -2.

$M_4 = \{ z \in \mathbb{C} \mid |z+2| > \frac{3}{2} \text{ and } |z-1| > \frac{3}{2} \}.$