Complex Functions: Auditorium Exercise-07

For Engineering Students

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Functions are often not analytic at individual points. Potential/electric field of a charge Q at the point z_0:

$$\Phi(z) = \frac{k \cdot Q}{\|z - z_0\|}, \qquad E(z) = \nabla\Phi(z) = \frac{k \cdot Q}{\|z - z_0\|^3}(z - z_0)$$

Behavior of the functions near such points:
z_0 is called an **isolated singularity** of the analytic function $f : G \to \mathbb{C}$, if a punctured neighborhood of z_0 belongs to G, but not z_0 itself:

$$0 < |z - z_0| < r \subset G, \quad z_0 \notin G, \quad r > 0\,.$$

Example a):

- 0 and 1 are isolated singularities of the function $f(z) := \dfrac{z-1}{z^2(z-1)}$.

Let z_0 be an isolated singularity of f. Then f can be expanded into a Laurent series in $0 < |z - z_0| < r$:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k .$$

- z_0 is called a **removable singularity** $\iff$ $a_k = 0 \quad \forall k < 0 .$
- z_0 is called a **pole of order** m with $m \in \mathbb{N}$ $\iff$ $a_{-m} \neq 0 , \quad a_k = 0 \quad \forall k < -m .$
  The Laurent series then has the form
  $$a_{-m} \frac{1}{(z - z_0)^m} + a_{-m+1} \frac{1}{(z - z_0)^{m-1}} + \cdots + a_{-1} \frac{1}{(z - z_0)^1} + \sum_{k=0}^{\infty} a_k (z - z_0)^k$$
- z_0 is called an **essential singularity** $\iff$ $a_k \neq 0$ for infinitely many $k < 0$.

Example b) $f(z) := \dfrac{1}{z(z^2+4)}$ The function has isolated singularities at the zeros of the denominator $0,\ 2i,\ -2i$.
For classification, there are (at least) three possibilities.
1st Possibility: Calculate the series for $z_0 = 0$ and $0 < |z| < 2$ we have

$$f(z) = \frac{1}{z} \cdot \frac{1}{z^2+4} = \frac{1}{z} \cdot \frac{1}{4(1+\frac{z^2}{4})} = \frac{1}{4z} \cdot \frac{1}{\left(1-(-\frac{z^2}{4})\right)}$$

$$= \frac{1}{4z} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} z^{2k} = \frac{1}{4z} + \underbrace{\sum_{k=1}^{\infty} \frac{(-1)^k}{4^{k+1}} z^{2k-1}}_{\text{positive powers}}$$

Thus, $z_0 = 0$ is a pole of the first order.

Example: Poles of Order One (Cont.)

2nd Possibility: Series is not calculated

$$f(z) = \frac{1}{z} \cdot \underbrace{\frac{1}{z^2+4}}_{g(z)}$$

$g(z)$ is holomorphic (analytic) near $z_0 = 0$ and can be expanded into a Taylor series

$$g(z) = g(z_0) + \sum_{k=1}^{\infty} \frac{g^{(k)}(z_0)}{k!} (z - z_0)^k$$

Thus, f near z_0 is given by

$$f(z) = \frac{g(0)}{z} + \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} (z - 0)^{k-1}$$

Since $g(0) = 1/4 \neq 0$, the lowest power of $(z - 0)$ appearing in the series is -1.

WARNING: For $|z| > 2$ one obtains for the same function f

$$f(z) = \frac{1}{z} \cdot \frac{1}{z^2+4} = \frac{1}{z} \cdot \left(\frac{1}{z^2\left(1+\frac{4}{z^2}\right)} \right) = \ldots = \sum_{k=-\infty}^{-1} (-4)^{-k-1}\, z^{2k-1}$$

Thus, infinitely many terms with negative powers!

Question: So is there actually an essential singularity at $z_0 = 0$?

Is $z = 0$ is a simple zero of the denominator, so a simple pole of f?

However: The behavior of f near z_0 is given by the Laurent series that converges to f in a punctured neighborhood of z_0.

Example c)

$$f(z) := \frac{\sin(z)}{z^4} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \ldots}{z^4}$$

Does f have a pole of order 4 at $z_0 = 0$?

First, write the series expansion of $\sin(z)$:

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \cdots$$

$$f(z) = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} \pm \cdots$$

Example: Essential Singularity

Example d) $f(z) := e^{\frac{1}{z+3}}$ has an isolated singularity at $z_0 = -3$.

To analyze the nature of this singularity, we expand $f(z)$ as a series:

$$f(z) = e^{\frac{1}{z+3}} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{z+3} \right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} (z - (-3))^{-k}$$

Here, the term $(z - (-3))^{-k}$ indicates that the series contains infinitely many negative powers of $z + 3$.

Example: Simple Poles

Example e) $f(z) := \dfrac{z+2i}{z^2+4}$

The function $f(z) = \dfrac{z+2i}{z^2+4}$ has two singularities:

- A simple pole at $z_1 = 2i$.
- A simple pole at $z_2 = -2i$.

WARNING: removable $\neq$ cancelable!

Example f): $f(z) := \dfrac{\cos(z) - 1}{z^2}$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Let z_0 be an isolated singularity of f and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

the Laurent series of f in $\underline{0 < |z - z_0| < r}$. Then,

$$h_f(z; z_0) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

is called the **principal part** of f at z_0.
It holds for any closed curve C with $\mathrm{Ind}(C, z_0) = 1$:

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

a_{-1} is the **residue of f at** $z_0 = \mathrm{Res}\, f(z_0) = \mathrm{Res}\,(f; z_0)$.

If $f(z)$ is analytic at z_0

$\implies \operatorname{Res}(f; z_0) = 0$

If z_0 is a removable singularity

$\implies \operatorname{Res}(f; z_0) = 0$

If z_0 is a pole of the first order

$\implies h_f(z; z_0) = \frac{a_{-1}}{(z-z_0)} = \frac{\operatorname{Res}(f; z_0)}{(z-z_0)}$

Partial Fraction Decomposition: A rational function that vanishes at infinity, is the sum of its principal parts!

This means: With $\quad f(z) = \dfrac{p(z)}{q(z)},$

p and q polynomials with $\mathrm{grad}(p) < \mathrm{grad}(q)$

$z_1, \cdots, z_k$ be the zeros of q,

one determines the principal part for each z_l, $l = 1, \cdots, k$ and obtains

$f(z) \;=\; h_f(z; z_1) \;+\; h_f(z; z_2) \;+\; \cdots + h_f(z; z_k)$

The complex partial fraction decomposition (PFD) of f.

Example : $g(z) = \dfrac{2+3z+z^2}{(z^2+4)(z^2-1)}$,

$$g(z) = \frac{2+3z+z^2}{(z^2+4)(z^2-1)} = \frac{(1+z)(2+z)}{(z^2+4)(z+1)(z-1)}.$$

g has a removable singularity at $z_0 = -1$ and simple poles $z_1 = 1,\ z_2 = 2i,\ z_3 = -2i$. Using their respective principal parts, we have

$$g(z) = h_f(z; z_1) + h_f(z; z_2) + h_f(z; z_3)$$

For $z_1 = 1$:

$$g(z) = \frac{1}{z-1} \frac{(1+z)(2+z)}{(z^2+4)(z+1)} = \frac{1}{z-1} k(z)$$

k is holomorphic in an appropriate neighborhood of 1. Thus,

$$h_g(z;1) = \frac{a_{-1}}{z-1} = \frac{k(1)}{z-1} = \frac{3}{5} \cdot \frac{1}{z-1}$$

For $z_2 = 2i$, we calculate

$$g(z) = \frac{1}{z-2i} \underbrace{\left[\frac{(1+z)(2+z)}{(z+2i)(z-1)(z+1)}\right]}_{\tilde{k}(z)}$$

$$= \frac{1}{z-2i} \left(\tilde{k}(2i) + \tilde{k}'(2i)(z-(-2i)) + \dots\right)$$

$$h_g(z;2i) = \frac{1}{z-2i} \left[\frac{(1+z)(2+z)}{(z+2i)(z-1)(z+1)}\right]_{z=2i} = -\frac{3+i}{10} \frac{1}{z-2i}$$

Similarly, for $z_3 = -2i$

$$h_g(z;-2i) = \frac{1}{z-2i}\left[\frac{(1+z)(2+z)}{(z+2i)(z-1)(z+1)}\right]_{z=-2i} = \frac{i-3}{10}\,\frac{1}{z+2i}$$

The complex partial fraction decomposition is therefore

$$g(z) = \frac{3}{5}\,\frac{1}{z-1} - \frac{3+i}{10}\,\frac{1}{z-2i} - \frac{3-i}{10}\,\frac{1}{z+2i}$$

- **Rule 1)** z_0 simple pole: (Enclosure method)
  $\mathrm{Res}\ f(z_0) = \lim_{z\to z_0}(z-z_0)f(z)$
- **Rule 2)** $f(z) = \frac{p(z)}{q(z)}$, p and q holomorphic in a neighborhood of z_0,
  z_0 simple zero of q, $p(z_0) \neq 0 \Longrightarrow$
  z_0 is a first-order pole of f with $\mathrm{Res}\ f(z_0) = \frac{p(z_0)}{q'(z_0)}$
- **Rule 3)** z_0 pole of order m
  $\mathrm{Res}\ f(z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left((z-z_0)^m f(z)\right)^{(m-1)}$

Let, $D \subset \mathbb{C}$ be a domain, $f : D \setminus \{z_1, \cdots, z_m\} \to \mathbb{C}$ holomorphic,

C a closed path in $D \setminus \{z_1, \cdots, z_m\}$, piecewise C^1. Then,

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Ind}(C, z_k) \operatorname{Res}(f; z_k)$$

In particular, if C is a simply closed, positively oriented curve and $z_1, \cdots, z_m$ are inside C (i.e., $\operatorname{Ind}(C, z_k) = 1$),

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res} f(z_k)$$

Calculation of Residues:

$$f(z) = \frac{z}{z^4 - 1} = \frac{z}{(z^2-1)(z^2+1)} = \frac{z}{(z-1)(z+1)(z-i)(z+i)}$$

Rule 1 for $z = i$

$$\operatorname{Res} f(i) = \lim_{z \to i} \left((z-i) f(z)\right) = \lim_{z \to i} \frac{(z-i)z}{(z-1)(z+1)(z-i)(z+i)}$$

$$= \left. \frac{z}{(z-1)(z+1)(z+i)} \right|_{z=i} = -\frac{1}{4}$$

Rule 2 for $z = -i$

$$\operatorname{Res} f(-i) = \left. \frac{z}{(z^4-1)'} \right|_{z=-i} = \left. \frac{z}{4z^3} \right|_{z=-i} = -\frac{1}{4}$$

$\operatorname{Res} f(-1) = \frac{1}{4}$ $\qquad$ $\operatorname{Res} f(1) = \frac{1}{4}$

Rule of Thumb for First-Order Poles: Rule 1) simpler if the denominator is presented as a product of terms. Rule 2) simpler if the denominator is fully expanded.

$$\oint_{C_1} f(z)\,dz = -2\pi i \,\mathrm{Res} f(i) = \frac{\pi i}{2}$$

$$\oint_{C_2} f(z)\,dz = +2\pi i \,[\mathrm{Res} f(-i) + \mathrm{Res}(-1)] = 0$$

$$\oint_{C_3} f(z)\,dz = -2\pi i \sum_{k=1}^{4} \mathrm{Res} f(z_k) = -2\pi i \left(\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4}\right) = 0$$

The solution involves calculating residues at the singularities of the function and then applying these residues to evaluate the integrals over different contours.

If f has only simple poles, then each principal part consists of a single term, namely

$$h_f(z; z_k) = \frac{\text{Res}(f; z_k)}{z - z_k}$$

Previously, we had:

$$f(z) = \frac{z}{z^4 - 1} = \frac{z}{(z-1)(z+1)(z-i)(z+i)}$$

Thus, we have:

$$f(z) = h_f(z; -i) + h_f(z; i) + h_f(z; -1) + h_f(z; 1).$$

With $\text{Res} f(\pm i) = -\frac{1}{4}, \quad \text{Res} f(\pm 1) = \frac{1}{4}$,
we immediately obtain the partial fraction decomposition:

$$f(z) = \sum_{k=1}^{4} \frac{\text{Res}(f; z_k)}{z - z_k}$$

$$= \frac{\frac{-1}{4}}{z+i} + \frac{\frac{-1}{4}}{z-i} + \frac{\frac{1}{4}}{z+1} + \frac{\frac{1}{4}}{z-1}.$$

Be cautious with the Cover-Up Method: For example, with

$$f(z) = \frac{1}{(z-1)(2z-4)}$$

it holds that $\operatorname{Res} f(2) \neq \frac{1}{(z-1)}\Big|_{z=2}$?

$$\operatorname{Res}(f,1) = \lim_{z\to 1}(z-1)\frac{1}{2(z-1)(z-2)} = \lim_{z\to 1}\frac{1}{2(z-2)} = \frac{1}{2(1-2)} = -\frac{1}{2}$$

$$\operatorname{Res}(f,2) = \lim_{z\to 2}(z-2)\frac{1}{2(z-1)(z-2)} = \lim_{z\to 2}\frac{1}{2(z-1)} = \frac{1}{2(2-1)} = \frac{1}{2}$$

Example for Rule 3)

$$f(z) = \frac{z^2-1}{(z-1)^2(z+1)(z-i)^4}$$

Example: Higher Order Pole Residue

Singularities:

$$\begin{cases} z_1 = 1 & \text{Pole of first order} \\ z_2 = i & \text{Pole of fourth order} \\ z_3 = -1 & \text{Removable singularity} \end{cases}$$

$$\implies \quad \mathrm{Res} f(z_3) = 0$$

$$\implies \quad \mathrm{Res} f(1) = \frac{1}{(1-i)^4}$$

$$Resf(i) = \frac{1}{3!} \lim_{z\to i} \left((z-i)^4 f(z)\right)''' = \frac{1}{6} \lim_{z\to i} \left((z-i)^4 \frac{1}{(z-1)(z-i)^4}\right)'''$$

$$= \frac{1}{6} \lim_{z\to i} \left(\frac{1}{z-1}\right)''' = -(z-1)^{-4}\Big|_{z=i}$$

$$= -\frac{1}{(i-1)^4} = -\frac{1}{(-2i)^2} = \frac{1}{4}\,.$$

Another Example:

$f(z) := \frac{\sin(z)}{z^4}$

$$f(z) = \frac{1}{z^4}\sin(z) = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \ldots\right)$$

$$= \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} \pm \ldots$$

Zero is a pole of third order!
Of course, one does NOT apply Rule 3) to calculate $\operatorname{Res} f(0)$. Instead?

The residue at $z = 0$ is:

$$\operatorname{Res} f(0) = -\frac{1}{3!} = -\frac{1}{6}$$

Theorem A) Let $H := \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$

D a domain with $H \subset D$.
$f(z)$ is holomorphic in $D \setminus \{z_1, \cdots, z_n\}$.
$\{z_1, \cdots, z_n\} \subset H \setminus \mathbb{R}$
(i.e., no singularities on $\mathbb{R}$),

$$\lim_{z \to \infty} z \cdot f(z) = 0 \text{ uniformly in } H.$$

(e.g., if $f(z) = \frac{p(z)}{q(z)}$ with grad q $\geq$ grad p $+2$)

Then $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f(z_k)$

Example 1) $\int_{-\infty}^{\infty} \frac{1}{(x-2)(x^2+1)}\,dx$

The theorem is not applicable! Singularity at $z = 2 \in \mathbb{R}$.

Example 2) $I = \int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+1)}\,dx$

$f(z) = \frac{1}{(z+2i)(z-2i)(z+i)(z-i)}$

Singularities in the upper half-plane: $z_1 = i, z_2 = 2i$

$I = 2\pi i\,(\mathrm{Res} f(i) + \mathrm{Res} f(2i))$

$$= 2\pi i \left(\left.\frac{1}{(z+2i)(z-2i)(z+i)}\right|_{z=i} + \left.\frac{1}{(z+2i)(z-i)(z+i)}\right|_{z=2i} \right)$$

Theorem B) Almost all conditions are as in Theorem A).
In particular: finitely many singularities $\{z_1, \cdots, z_n\}$ in $H \setminus \mathbb{R}$.
Different from A):

$$\lim_{z\to\infty,\,\mathrm{Im}(z)\geq 0} f(z) = 0 \qquad , z = x + iy$$

(e.g., $f(z) = \frac{p(z)}{q(z)}$, Grad (q) > Grad (p)).

then for $\omega > 0$,

$$\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx \;=\; 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(e^{i\omega z} f; z_k\right)$$

(compare with formulas for Fourier transform. There:

$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega\,, \quad F(\omega) = \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}\,d\tau\,)$

THANK YOU