Fachbereich Mathematik der Universität Hamburg **SoSe 2025**
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Complex Functions for student of engineering sciences

Exam SoSe 25 - Solutions

Problem 1. (5 points)

Let

$$M = \left\{ z = re^{i\varphi} \in \mathbb{C} \;\middle|\; r \in [e, e^2],\ \varphi \in \left[-\frac{\pi}{4}, 0\right] \right\}$$

and

$$f(z) = \operatorname{Log}\left(e^{i\cdot\pi/4} \cdot z^2\right),$$

where Log denotes the principal branch of the natural logarithm.

(a) Determine the image $f(M)$ of the set M under the function f.

(b) Sketch the sets M and $f(M)$, or describe them with words.

Solution.

(a) We decompose

$$f = f_3 \circ f_2 \circ f_1$$

with

$$u = f_1(z) = z^2, \qquad v = f_2(u) = e^{i\cdot\frac{\pi}{4}} \cdot u, \qquad w = f_3(v) = \operatorname{Log}(v). \qquad (1\text{ P.})$$

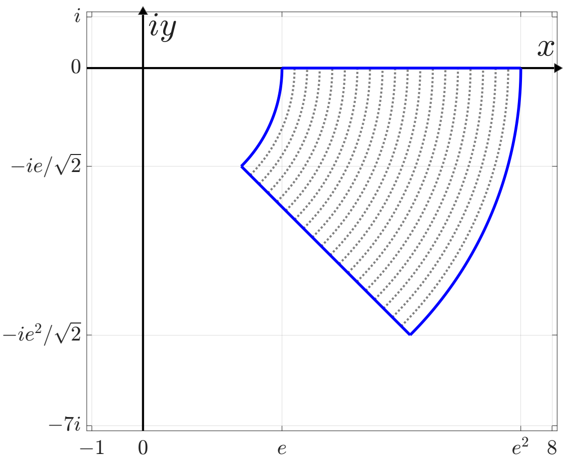
Then

$$U = f_1(M) = \left\{ z = re^{i\varphi} \in \mathbb{C} \;\middle|\; r \in [e^2, e^4],\ \varphi \in \left[-\frac{\pi}{2}, 0\right] \right\}, \qquad (1\text{ P.})$$

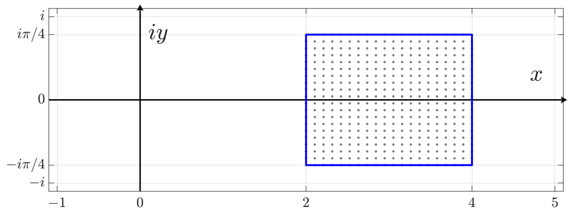
$$V = f_2(U) = \left\{ z = re^{i\varphi} \in \mathbb{C} \;\middle|\; r \in [e^2, e^4],\ \varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \right\}, \qquad (1\text{ P.})$$

$$W = f_3(V) = f(M) = \left\{ z = x + iy \in \mathbb{C} \;\middle|\; x \in [2, 4],\ y \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \right\}. \qquad (1\text{ P.})$$

(b)



M



$f(M)$

M : Segment of the ring around zero with inner radius e, outer radius e^2, and angles between $-\pi/4$ and 0 (including the boundary).

$f(M)$: Rectangle with edges form 2 to 4 and from $-\pi/4$ to $\pi/4$ (including the boundary). (1 P.)

Problem 2. (5 points)

For $z = x + \mathrm{i}y \in \mathbb{C}$ with $x, y \in \mathbb{R}$ let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be given by

$$f(z) = \frac{1}{2}(x-y)^2 + \mathrm{i} \cdot \left(\frac{1}{2}x^2 - xy\right).$$

(a) Determine all points in which f is complex differentiable

(b) Is f conformal in $z_0 = 0$?

Solution.

(a) With

$$u(x,y) = \frac{1}{2}(x-y)^2, \qquad v(x,y) = \left(\frac{1}{2}x^2 - xy\right)$$

we have

$$\begin{aligned} u_x(x,y) &= x - y, & u_y(x,y) &= y - x, \\ v_x(x,y) &= x - y, & v_y(x,y) &= -x. \end{aligned} \qquad \text{(2 P.)}$$

From the Cauchy-Riemann differential equations we get

$$u_x(x,y) \stackrel{!}{=} v_y(x,y) \quad \Rightarrow \quad x - y = -x \quad \Rightarrow \quad y = 2x. \qquad \text{(1 P.)}$$

The second equation,

$$-u_y(x,y) = x - y = v_x(x,y),$$

is satisfied for all $x, y \in \mathbb{R}$. Thus, f is differentiable in all points of the form $z = \alpha(1 + 2\mathrm{i})$, $\alpha \in \mathbb{R}$. (1 P.)

(b) The functions u, v are continuously differentiable on $\mathbb{R}^2$. Therefore, in the case of conformity, we would have $f'(0) \neq 0$. But

$$f'(0) = u_x(0,0) + \mathrm{i}v_x(0,0) = 0,$$

and thus f is not conformal in $z_0 = 0$. (1 P.)

Problem 23. (5 points)

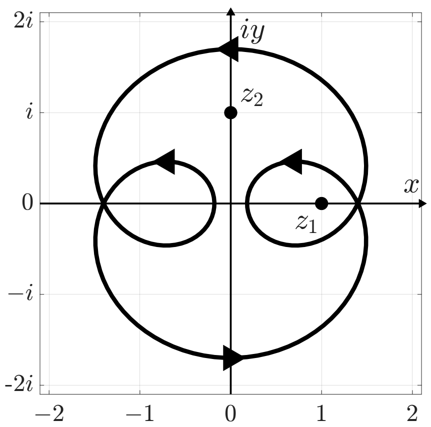
Consider the function

$$f(z) \;=\; \frac{1}{(z-1)(z-\mathrm{i})}.$$

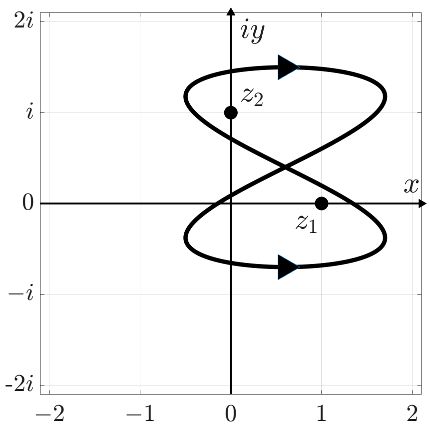
(a) Determine and classify all isolated singularities of f.
Compute the corresponding residues.

(b) The following figures show closed curves Γ. We assume that these curves are passed through once and that the arrows indicate their orientation.

Determine $\displaystyle\int_\Gamma f(z)\,\mathrm{d}z$ for both curves, respectively.



(1)



(2)

Solution.

(a) The isolated singularities lie at $z_1 = 1$ and $z_2 = \mathrm{i}$. Both are simple poles of f. (1 P.)
We find

$$\begin{aligned}
\mathrm{Res}(f,1) &= \left.\frac{1}{z-\mathrm{i}}\right|_{z=1} = \frac{1}{1-\mathrm{i}} = \frac{1}{2}(1+\mathrm{i}),\\
\mathrm{Res}(f,\mathrm{i}) &= \left.\frac{1}{z-1}\right|_{z=\mathrm{i}} = \frac{1}{\mathrm{i}-1} = -\frac{1}{2}(1+\mathrm{i}).
\end{aligned}$$

(2 P.)

(b) The curve in (1) goes around $z_1 = 1$ twice in positive direction and once around $z_2 = \mathrm{i}$ in positive direction. From the residue theorem we get:

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\cdot(2\mathrm{Res}(f,1)+\mathrm{Res}(f,\mathrm{i})) = 2\pi\mathrm{i}\cdot\frac{1}{2}(1+\mathrm{i}) = \pi(-1+\mathrm{i}). \quad \text{(1 P.)}$$

The curve in (2) goes around $z_1 = 1$ once in positive direction and around $z_2 = \mathrm{i}$ once in negative direction:

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\cdot(\mathrm{Res}(f,1)-\mathrm{Res}(f,\mathrm{i})) = 2\pi\mathrm{i}\cdot(1+\mathrm{i}) = 2\pi(-1+\mathrm{i}). \quad \text{(1 P.)}$$

Problem 4. (5 points)

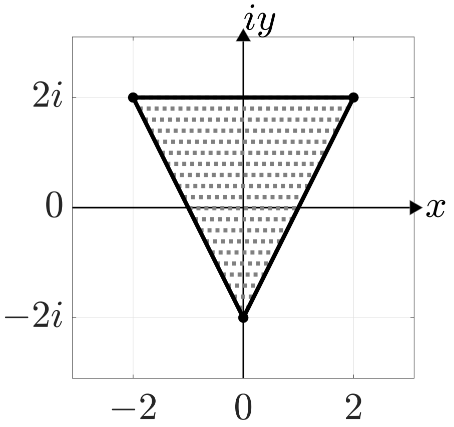
(a) Determine the Möbius transformation $T : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ with

$$T(-2\mathrm{i}) = \infty, \qquad T(-\mathrm{i}) = 4, \qquad T(3\mathrm{i}) = 0.$$

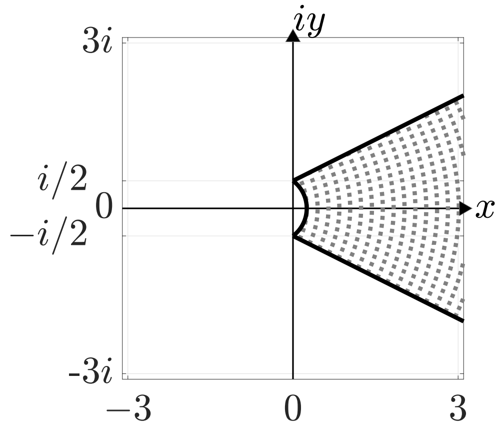
(b) Determine the image of the imaginary axis under T.

(c) Let M be the interior of the triangle with corners
$z_0 = -2\mathrm{i}, \quad z_1 = -2 + 2\mathrm{i}, \quad z_2 = 2 + 2\mathrm{i}$ (see figure below).

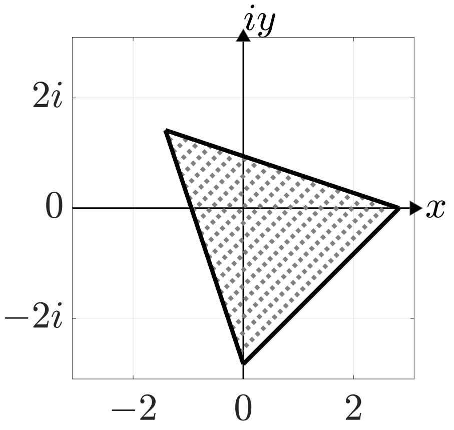
Which of the following figures shows the image of M under T?
Explain your answer.



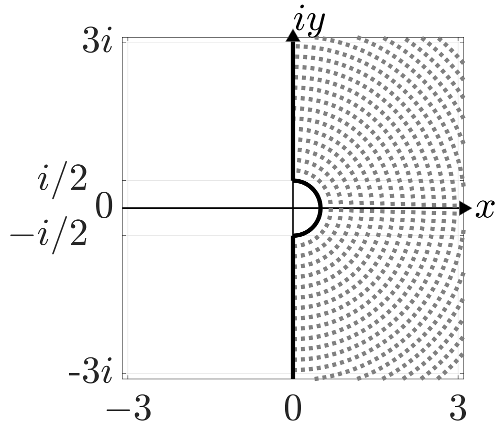
M



(1)



(2)



(3)

Lösung.

(a) We know the zero and the pole of T, so we can use the anssatz

$$T(z) = \alpha \cdot \frac{z - 3\mathrm{i}}{z + 2\mathrm{i}}, \qquad \alpha \in \mathbb{C}.$$

(1 P.)

From the remaining condition we get

$$T(-\mathrm{i}) \;=\; \alpha \cdot \frac{-\mathrm{i} - 3\mathrm{i}}{-\mathrm{i} + 2\mathrm{i}} = \alpha \cdot \frac{-4\mathrm{i}}{\mathrm{i}} \;=\; \alpha \cdot (-4) \stackrel{!}{=} 4 \qquad \Rightarrow \qquad \alpha \;=\; -1,$$

and therefore $T(z) = \dfrac{3\mathrm{i} - z}{2\mathrm{i} + z}$. (1 P.)

(b) Since $-d/c = -2\mathrm{i} \in \mathrm{i}\mathbb{R}$, the image of $\mathrm{i}\mathbb{R}$ is a straight line. We have $T(-\mathrm{i}) = 4 \in \mathbb{R}$ and $T(3\mathrm{i}) = 0 \in \mathbb{R}$.It follows that $T(\mathrm{i}\mathbb{R}) = \mathbb{R}$. (1 P.)

(c) The line through z_1 and z_2 does not pass through $-2\mathrm{i}$, so its image is a (true) circle. The image of M thus cannot be bounded by three straight line. This rules out (2).

Alternatively: The point $-2\mathrm{i}$ is mapped to ∞. Therefore, the image of M cannot be bounded. (1 P.)

In z_1 and z_2 the edges of the triangle intersect at an angle smaller than $\pi/2$. Möbius-Transformationen are angle-preserving, which rules out (3). All angles in (3) are $\pi/2$. (1 P.)

The correct image is therefore (1).