Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
Prof. Dr. J. Struckmeier,
Dr. H. P. Kiani, T. Hassan

Complex functions for Engineering Students

Sheet 7 (Homework)

Exercise 1:

Determine and classify all isolated singularities of the following functions.

a) $f(z) = z^3 \cdot \sinh(\frac{1}{z})$,

b) $f(z) = \dfrac{\sin(z) - z}{z^2(\frac{\pi^2}{4} - z^2)}$,

c) $f(z) = \dfrac{\ln(z)}{(z-1)^4}, \qquad z \in \mathbb{C} \setminus (-\infty, 0]$.

Solution Hints for Exercise 1:

Here in parts a) and b), the known series of sin and sinh are used. Of course, one can also argue as in part c) or as done in the presence exercises for the function s .

a) For $f(z) = z^3 \cdot \sinh(\frac{1}{z}) = z^3 \displaystyle\sum_{k=0}^{\infty} \frac{z^{-2k-1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{z^{-2k-1}}{(2k-2)!}$

there is an essential singularity at $z_0 = 0$.

b)

$$f(z) = \frac{\sin(z) - z}{z^2(\frac{\pi^2}{4} - z^2)} = \frac{1}{(\frac{\pi}{2} - z)(\frac{\pi}{2} + z)} \cdot \frac{\left(\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}\right) - z}{z^2}$$

$$= \frac{1}{(\frac{\pi}{2} - z)(\frac{\pi}{2} + z)} \cdot \frac{\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}}{z^2} = \frac{1}{(\frac{\pi}{2} - z)(\frac{\pi}{2} + z)} \cdot \sum_{k=1}^{\infty} (-1)^k \frac{z^{2k-1}}{(2k+1)!}$$

Thus, there is a removable singularity at $z_0 = 0$. Due to $\sin(\pm\frac{\pi}{2}) \neq \pm\frac{\pi}{2}$, there are first-order poles at $z_{1,2} := \pm\frac{\pi}{2}$.

c) $f(z) = \dfrac{\ln(z)}{(z-1)^4}$

To compute the Taylor expansion of $g(z) := \ln(z)$ around the point $z_0 := 1$, we have

$\ln(1) = 0$,

$\ln(z)' = \frac{1}{z} \implies (\ln(z))'_{z=1} = 1$.

Thus, the Taylor expansion of $g(z) = \ln(z)$ around $z_0 := 1$ is

$$T_g(z;1) = \ln(1) + (z-1) + \sum_{n=2}^{\infty} \frac{g^{(n)}(1)}{n!}(z-1)^n$$

and for $f(z) = \dfrac{g(z)}{(z-1)^4}$, we obtain:

$$f(z) = \frac{1}{(z-1)^3} + \sum_{n=2}^{\infty} \frac{g^{(n)}(1)}{n!}(z-1)^{n-4}.$$

Thus, there is a third-order pole.

Exercise 2:

Calculate the following integrals or their (Cauchy) principal values with the use of the residue (see lecture notes pages 149-151).

a) $\displaystyle\int_0^\infty \frac{1}{x^4+16}\,dx.$

b) $\displaystyle\int_{-\infty}^\infty \frac{x\cos(\omega x)}{x^2+4}\,dx \qquad \omega > 0.$

c) $\displaystyle\int_{-\infty}^\infty \frac{x\sin(\omega x)}{x^2+4}\,dx \qquad \omega > 0.$

Solution:

a) The roots of the equation $z^4+16=0$ are the fourth roots of $16e^{i\pi}$, thus

$$z_1 = 2e^{i\frac{\pi}{4}}, \quad z_2 = 2e^{i\frac{3\pi}{4}}, \quad z_3 = 2e^{i\frac{5\pi}{4}}, \quad z_4 = 2e^{i\frac{7\pi}{4}},$$

Only the first two lie in the upper half-plane. Therefore, we obtain

$$\int_{-\infty}^\infty \frac{1}{x^4+16}\,dx \;=\; 2\pi i(Res\, f(z_1) + Res\, f(z_2))$$

For $f(z) := \dfrac{1}{z^4+16}$,

$$Res\, f(z_k) \;=\; \frac{1}{4z_k^3}.$$

Thus,

$$\begin{aligned}\int_0^\infty \frac{1}{x^4+16}\,dx &= \pi i\frac{1}{32}\left(e^{-i\frac{3\pi}{4}} + e^{-i\frac{9\pi}{4}}\right) \\ &= \frac{\pi i}{32}\left(e^{-i\frac{3\pi}{4}} + e^{-i\frac{\pi}{4}}\right) \;=\; \frac{\pi i}{32}2i\sin(-\frac{\pi}{4}) \\ &= \frac{\pi\sqrt{2}}{32}\end{aligned}$$

b) $\displaystyle\int_{-\infty}^\infty \frac{x\cos(\omega x)}{x^2+4}\,dx = 0.$

because the integrand is odd! This result is also obtained in part c).

c)

$$\int_{-\infty}^\infty \frac{x\sin(\omega x)}{x^2+4}\,dx \qquad \omega > 0.$$

The function $f(z) = \dfrac{z}{z^2+4}$ has isolated singularities at $\pm 2i$. None of these singularities lie on the real axis. We have $\lim\limits_{|z|\to\infty} f(z) = 0$. Only $2i$ lies in the upper half-plane. According to the lecture, we obtain

$$\begin{aligned} \mathrm{PV} \int_{-\infty}^{\infty} \frac{x e^{i\omega x}}{x^2+4}\,dx &= 2\pi i Res\left(\frac{z e^{i\omega z}}{z^2+4}\right)\bigg|_{z=2i} \\ &= 2\pi i \left(\frac{2i e^{i\omega 2i}}{2\cdot 2i}\right) = i\pi e^{-2\omega} \end{aligned}$$

and consequently

$$\int_{-\infty}^{\infty} \frac{x\sin(\omega x)}{x^2+4}\,dx \;=\; \mathrm{Im}\left(i\pi e^{-2\omega}\right) \;=\; \pi e^{-2\omega}$$

Note: The result from b) is also obtained by calculation here, namely

$$\int_{-\infty}^{\infty} \frac{x\cos(\omega x)}{x^2+4}\,dx \;=\; \mathrm{Re}\left(i\pi e^{-2\omega}\right) \;=\; 0.$$

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