Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
Prof. Dr. J. Struckmeier,
Dr. H. P. Kiani, T. Hassan

Complex functions for Engineering Students

Work sheet 6

Exercise 1: Calculate the following integrals:

a) $\displaystyle\oint_{C_1} \frac{e^z}{z}\,dz,$ $\quad C_1 : [0, 2\pi] \to \mathbb{C}, \quad C_1(t) = 2 + e^{it},$

b) $\displaystyle\oint_{C_2} \frac{e^z}{z}\,dz\,,$ $\quad C_2 : [0, 2\pi] \to \mathbb{C},\ C_2(t) = 2\,e^{it},$

c) $\displaystyle\oint_{C_2} \frac{\pi e^{iz^2}}{(z-i)^2}\,dz$ $\quad C_2 : [0, 2\pi] \to \mathbb{C},\ C_2(t) = 2\,e^{it},$

d) $\displaystyle\oint_{C_3} \frac{z\cos(2z)}{(z-\frac{\pi}{3})^3}\,dz$ $\quad C_3 : [0, 2\pi] \to \mathbb{C},\ C_3(t) = 1 + e^{it},$

e) $\displaystyle\oint_{C_4} \frac{z\cos(2z)}{(z-\frac{\pi}{3})^3}\,dz$ $\quad C_4 : [0, 6\pi] \to \mathbb{C},\ C_4(t) = \frac{1}{2}\,e^{2it},$

f) $\displaystyle\oint_{C_5} \frac{z\cos(2z)}{(z-\frac{\pi}{3})^3}\,dz$ $\quad C_5 : [0, 6\pi] \to \mathbb{C},\ C_5(t) = 1 + 2e^{it},$

g) $\displaystyle\oint_{C_6} \frac{1}{z^2+2z+10}\,dz\,,$ $\quad C_6 : [0, 4\pi] \to \mathbb{C},\ C_6(t) = -3i + 3\,e^{-it}.$

h) $\displaystyle\oint_{C_7} \frac{z^2+2}{(z^3-z^2+z-1)}\,dz,$ $\quad C_7 : |z - 0.5| = 1$, traversed once counterclockwise.

Solution to Exercise 1:

a) According to Cauchy's Integral Theorem, we have: $\displaystyle\oint_{C_1} \frac{e^z}{z}\,dz = 0.$

b) According to the first Cauchy's Integral Formula (see lecture for conditions):

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0).$$

Therefore, $\displaystyle\oint_{C_2} \frac{e^z}{z}\,dz = 2\pi i e^0 = 2\pi i$

c) According to the second Cauchy's Integral Formula (see lecture for conditions):

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \;=\; 2\pi i \frac{f^{(n)}(z_0)}{n!}$$

Therefore, $$\oint\limits_{C_2} \frac{\pi e^{iz^2}}{(z-i)^2} \;=\; 2\pi i \left(\pi e^{iz^2}\right)'_{z=i} \;=\; 2\pi^2 i^2 \cdot 2ie^{i^3} \;=\; -4\pi^2 i e^{-i}\,.$$

d) Since $(z\cos(2z))'' \;=\; (-2z\sin(2z)+\cos(2z))' \;=\; -4z\cos(2z) - 4\sin(2z)\,,$

we obtain with the second Cauchy's Integral Formula:

$$I_3 := \oint\limits_{C_3} \frac{z\cos(2z)}{(z-\frac{\pi}{3})^3}\,dz = \;\pi i\,[-4z\cos(2z) - 4\sin(2z)]_{z=\frac{\pi}{3}} \;= 2\pi i(\frac{\pi}{3} - \sqrt{3})$$

e) From Cauchy's Integral Theorem, it follows that $\oint\limits_{C_4} \frac{z\cos(2z)}{(z-\frac{\pi}{3})^3}\,dz \;=\; 0\,.$

f) C_5 encircles the pole in the same manner as C_3 but three times. Therefore, we get:

$$I_5 := \oint\limits_{C_5} \frac{z\cos(2z)}{(z-\frac{\pi}{3})^3}\,dz = \;3I_3 = \;2\pi i(\pi - 3\sqrt{3})\,.$$

g) Consider $f(z) = \dfrac{1}{z^2+2z+10}\,.$

Finding the roots of the denominator: $z^2+2z+10 = (z+1)^2+9 = 0 \iff z_{1,2} = -1 \pm 3i\,.$

Then, $$\oint_{C_6} f(z)dz \;=\; \oint_{C_6} \frac{\frac{1}{z-(-1+3i)}}{z-(-1-3i)}\,dz \;= -2\cdot 2\pi i \cdot \frac{1}{-1-3i-(-1+3i)} \;=\; \frac{2\pi}{3}.$$

h) We have $\dfrac{z^2+1}{(z^3-z^2+z-1)} = \dfrac{z^2+1}{(z^2+1)(z-1)}\,.$

The points $\pm i$ are outside $|z-0.5| \le 1$. Thus, it follows that:

$$\oint\limits_{|z-0.5|=1} \frac{z^2+2}{(z^3-z^2+z-1)}\,dz = \oint\limits_{|z-0.5|=1} \frac{\frac{z^2+2}{z^2+1}}{(z-1)}\,dz = 2\pi i\left(\frac{1^2+2}{1^2+1}\right) = 3\pi i\,.$$

(Cauchy's Integral Formula with $f(z) = \frac{z^2+2}{z^2+1}$)

Exercise 2:

Given the function

$$g(z) = \frac{2+3z+z^2}{(z^2+4)(z^2-1)}, \qquad f(z) = \frac{1+z}{z^2(z+i)}, \qquad \tilde{f}(z) = \frac{\cos(z)-2}{z^2}:$$

a) How many Laurent series are there for g, f or $\tilde{f}$ at $z_0 = 0$?

b) Determine the Laurent series of the functions f and $\tilde{f}$ for the point $z_0 = 0$ that converges to $f(2)$ or $\tilde{f}(2)$ in the neighborhood of $z^* = 2$.

Hint: polynomial long division should be used!

Solution:

a) For g, there are three Laurent series around zero in the following rings:

$R_1 : 0 < |z| < 1, \qquad R_2 : 1 < |z| < 2, \qquad R_3 : 2 < |z|.$

For f, there are two Laurent series around zero in the following rings:

$R_1 : 0 < |z| < 1, \qquad R_2 : 1 < |z|.$

For $\tilde{f}$, there is only one Laurent series around zero. The series converges in the dotted complex plane:

$\mathbb{C} \setminus \{0\} : 0 < |z|.$

b) We approximate for $|z| > 1$ and $z_0 = 0$:

$$\begin{aligned} f(z) &= i + \frac{1+z}{z^2(z+i)} = \frac{1}{z^2} \cdot \frac{1+z}{z+i} = i + \frac{1}{z^2} \cdot \left(1 + \frac{1-i}{z+i}\right) \\ &= \frac{1}{z^2} + \frac{1-i}{z^2} \cdot \left(\frac{1}{z+i}\right) = \frac{1}{z^2} + \frac{1-i}{z^2} \cdot \left(\frac{1}{z} \cdot \frac{1}{1-\frac{-i}{z}}\right) \\ &= \frac{1}{z^2} + \frac{1-i}{z^3} \cdot \left(\sum_{k=0}^{\infty} \frac{(-i)^k}{z^k}\right) = \frac{1}{z^2} + \sum_{k=0}^{\infty} \frac{1-i}{i^k \cdot z^{k+3}} \\ &= z^{-2} + \sum_{k=-\infty}^{-3} (1-i) i^{k+3} z^k \end{aligned}$$

For $\tilde{f}$, it holds that in the dotted complex plane $0 < |z|$:

$$\tilde{f}(z) = \frac{\cos(z)-2}{z^2} = -\frac{2}{z^2} + \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} = -\frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} z^{2(k-1)}.$$

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