Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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Complex functions for Engineering Students

Sheet 6 (Homework)

Exercise 1:

a) Let C be the unit circle $|z| = 1$ traversed once in the mathematically positive direction.

(i) Compute $\displaystyle\int_C \frac{1}{(e^z - i)}\, dz\,.$

(ii) For a function analytic on $\mathbb{C}$, it is given that $|\, f(z)\,| = 4$ everywhere on the curve C and $f(0) = 4i$. What must f look like?

b) Let C be a smooth closed piecewise C^1 curve without double points. When is the integral

$$I(C) := \int_C \frac{z}{z^2+1}\, dz$$

defined?

What values can the integral take if it is defined?

Solution:

a) (i) $e^z = i \iff e^x e^{iy} = i \iff x = 0$ and $y = 2k\pi + \pi/2$.
Since $|2k\pi+\pi/2| \geq \pi/2$, the integrand is analytic in $|z| \leq 1 < \pi/2$. Therefore,

$$\int_{|z|=1} \frac{1}{(e^z - i)}\, dz \;=\; 0.$$

(ii) According to the lecture (maximum principle), f is constant, so $f(z) = 4i$.

b) The integral exists provided the curve does not pass through i or $-i$. The curve is simply closed (without double points), so the points i and $-i$ are not encircled multiple times.

If the curve is positively oriented, the following four cases can occur:

(i) Neither of the points $\pm i$ is encircled by the curve. Then, by Cauchy's integral theorem,

$$I(C) = 0$$

(ii) The point $-i$ is encircled by the curve, but not the point i. Then,

$$I(C) = \int_C \frac{\frac{z}{z-i}}{z+i}\, dz = 2\pi i \left[\frac{z}{z-i}\right]_{z=-i} = 2\pi i\, \frac{-i}{-2i} = \pi i$$

(iii) The point i is encircled by the curve, but not the point $-i$. Then,

$$I(C) = \int_C \frac{\frac{z}{z+i}}{z-i}\, dz = 2\pi i \left[\frac{z}{z+i}\right]_{z=i} = 2\pi i \frac{i}{2i} = \pi i$$

(iv) Both points are encircled. Using partial fraction decomposition,

$$\frac{z}{z^2+1} = \frac{1}{2}\left(\frac{1}{z+i} + \frac{1}{z-i}\right)$$

$$I(C) = \frac{1}{2}\int_C \frac{1}{z+i}\, dz + \frac{1}{2}\int_C \frac{1}{z-i}\, dz = \frac{1}{2}2\pi i + \frac{1}{2}2\pi i = 2\pi i$$

For a negatively oriented curve, the corresponding values are 0, $-\pi i$, $-2\pi i$.

Exercise 2: Determine the Laurent series for the following functions at the point of expansion z_0, which converges to $f(-3/2)$ at the point $z = -3/2$..

a) $f(z) = z^3 \cos(\frac{1}{z})$, $z_0 = 0$,

b) $f(z) = \dfrac{z^2+1}{z^2+z-2}$, $z_0 = 0$,

c) $f(z) = \dfrac{3}{z^2+z-2}$, $z_0 = 1$,

d) $f(z) = \dfrac{1}{(z-i)^3}$, $z_0 = 1+i$.

Solution hints for Exercise 2:

a)

$$\begin{aligned} f(z) &= z^3 \cos\left(\frac{1}{z}\right) \\ &= z^3\left(1 - \frac{1}{2z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \cdots\right) = \left(z^3 - \frac{z}{2} + \frac{1}{4!z} - \frac{1}{6!z^3} + \cdots\right) \end{aligned}$$

The function is analytic in $0 < |z| < \infty$. The Laurent series is

$$\begin{aligned} f(z) &= z^3 - \frac{z}{2} + \frac{1}{4!z} - \frac{1}{6!z^3} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{-2k+3} = \sum_{k=-\infty}^{1} \frac{(-1)^{k+1}}{(2-2k)!} z^{2k+1} \end{aligned}$$

b)

$$f(z) = \frac{z^2+1}{z^2+z-2} = 1 + \frac{-z+2+1}{(z+2)(z-1)} = 1 + \frac{1}{3}\left(\frac{2}{z-1} - \frac{5}{z+2}\right)$$

The function is analytic everywhere in $\mathbb{C}$ except at the points $z = 1$ and $z = -2$. Therefore, we consider the function on the ring $R_2 : 1 < |z| < 2$.

First, we determine the expansions of the individual terms:

$$|z| > 1 : \frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{k=0}^{\infty} z^{-k} = \sum_{k=-\infty}^{-1} z^k,$$

$$|z| < 2 : \frac{1}{z+2} = \frac{1}{2} \cdot \frac{1}{1-(\frac{-z}{2})} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-z)^k}{2^k} = -\sum_{k=0}^{\infty}(-\frac{1}{2})^{k+1} z^k$$

Thus, we obtain the following expansion of the function f in the ring $R_2 \ : \ 1 < |z| < 2$:

$$f(z) = 1 + \frac{2}{3}\left(\sum_{k=-\infty}^{-1} z^k + \frac{5}{3}\sum_{k=0}^{\infty}(-\frac{1}{2})^{k+1} z^k\right) = \left(\sum_{k=-\infty}^{-1} \frac{2}{3} z^k + \frac{1}{6} + \sum_{k=1}^{\infty} \frac{5}{3}(-\frac{1}{2})^{k+1} z^k\right)$$

Alternative:

$$f(z) = \frac{z^2+1}{z^2+z-2} = \frac{z^2+1}{(z+2)(z-1)} = \frac{z^2+1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right)$$

The function is analytic everywhere in $\mathbb{C}$ except at the points $z = 1$ and $z = -2$. Therefore, we consider the function on the ring $R_2 \ : \ 1 < |z| < 2$.

First, we determine the expansions of the individual terms:

$$|z| > 1 \ : \frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{k=0}^{\infty} z^{-k} = \sum_{k=-\infty}^{-1} z^k,$$

$$|z| < 2 \ : \frac{1}{z+2} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{z}{2})} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-z)^k}{2^k} = -\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{k+1} z^k$$

Thus, we obtain the following expansion of the function f in the ring $R_2 \ : \ 1 < |z| < 2$:

$$f(z) = \frac{z^2+1}{3}\left(\sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{k+1} z^k\right)$$

Combining terms gives

$$\begin{aligned}
f(z) &= \frac{1}{3}\left(\sum_{k=-\infty}^{-1} z^{k+2} + \sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{k+1} z^{k+2} + \sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{k+1} z^k\right) \\
&= \frac{1}{3}\left(\sum_{k=-\infty}^{-1} 2z^k + z^0 + z^1 + \sum_{k=2}^{\infty}\left[\left(-\frac{1}{2}\right)^{k-1} z^k + \left(-\frac{1}{2}\right)^{k+1} z^k\right] - \frac{1}{2}z^0 + \frac{1}{4}z^1\right) \\
&= \frac{1}{3}\left(2\sum_{k=-\infty}^{-1} z^k + \frac{1}{2} + \frac{5}{4}z + 5\sum_{k=2}^{\infty}\left(-\frac{1}{2}\right)^{k+1} z^k\right).
\end{aligned}$$

c) $f(z) = \dfrac{3}{z^2+z-2} = \dfrac{1}{z-1} - \dfrac{1}{z+2}, \qquad z_0 = 1$

is analytic everywhere in $\mathbb{C}$ except at the points $z = 1$ and $z = -2$. Therefore, we consider the function on the ring $R \ : \ 0 < |z-1| < 3$. Here,

$$\frac{1}{z+2} = \frac{1}{(z-1)+3} = \frac{1}{3} \cdot \frac{1}{1-\left(-\frac{z-1}{3}\right)} = \frac{1}{3}\sum_{k=0}^{\infty}\left(-\frac{z-1}{3}\right)^k.$$

Thus,

$$f(z) = \frac{1}{z-1} - \sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}}(z-1)^k.$$

d) The function $f(z) = \dfrac{1}{(z-i)^3}$ has an isolated singularity at $z_1 = i$. There is a Laurent series in the punctured disk $0 < |z - z_0| < 1$ and one in the annulus $1 < |z - z_0|$. First, we develop $g(z) = \dfrac{1}{z-i}$ in the outer annulus around $z_0 = 1+i$.

$$g(z) = \frac{1}{z-i} = \frac{1}{z-(1+i)+1} = \frac{1}{z-(1+i)} \cdot \frac{1}{1+\dfrac{1}{z-(1+i)}}$$

$$= \frac{1}{z-(1+i)} \sum_{k=0}^{\infty} \left(-\frac{1}{z-(1+i)}\right)^k = \sum_{k=0}^{\infty} (-1)^k \, (z-(1+i))^{-(k+1)}$$

$$= \sum_{k=-\infty}^{-1} (-1)^{k+1} \, (z-(1+i))^k$$

The desired series expansion is obtained by differentiating twice:

$$g'(z) = -\frac{1}{(z-i)^2} = \sum_{k=-\infty}^{-1} (-1)^{k+1} \, k \, (z-(1+i))^{k-1}$$

$$g''(z) = 2\frac{1}{(z-i)^3} = \sum_{k=-\infty}^{-1} (-1)^{k+1} \, k(k-1) \, (z-(1+i))^{k-2} \qquad \text{thus,}$$

$$\frac{1}{(z-i)^3} = \frac{1}{2} \sum_{k=-\infty}^{-1} (-1)^{k+1} \, k(k-1) \, (z-(1+i))^{k-2}$$

$$= \frac{1}{2} \sum_{k=-\infty}^{-3} (-1)^{k+1} \, (k+2)(k+1) \, (z-(1+i))^k$$

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