Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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Complex functions for Engineering Students

Work sheet 5

Exercise 1:

Calculate the following line integrals and sketch the corresponding curves.

a) $\displaystyle\int_{C_1+C_2} |z|\,dz := \int_{C_1} |z|\,dz + \int_{C_2} |z|\,dz,$ C_1 : straight path from -1 to 1, C_2 : half circle with radius 1 around the origin, from 1 to -1 traversed mathematically positive.

b) $\displaystyle\int_C (1+z)\,dz,$ $C(t) := \cos t + 3i\sin t,\ t \in [-\pi, 0]$ (half ellipse)

c) $\displaystyle\int_c (\bar{z})^2\,dz,$ $c(t) = 2e^{(-1+i)t},\ t \in [0,\ \pi/4],$

d) $\displaystyle\int_C e^{3z}\,dz,$ C : piece of the parabola $\operatorname{Im}(z) = \pi\,(\operatorname{Re}(z))^2$ that connects the points zero and $1+i\pi$.

Solution:

a) $\displaystyle\int_{C_1+C_2} |z|\,dz \qquad C_1 : t \mapsto t,\ t \in [-1,1], \quad C_2 : t \mapsto e^{it},\ t \in [0,\pi]$

$$\int_{C_1} |z|\,dz + \int_{C_2} |z|\,dz = \int_{-1}^{1} |t|dt + \int_0^{\pi} |e^{it}| ie^{it} dt$$

$$= \int_{-1}^{0} -tdt + \int_0^1 tdt + \left[e^{it}\right]_0^{\pi} = \left[t^2\right]_0^1 - 2 = -1$$

b) $$\int_C (1+z)\,dz = \int_{C(-\pi)}^{C(0)} (1+z)\,dz = \left[z + \frac{z^2}{2}\right]_{-1}^{1} = 2$$

Of course, $f(c(t))\dot{c}(t)$ can also be inserted into the line integral. This requires more work, however.

c) $c(t) = 2e^{(-1+i)t}, \quad \dot{c}(t) = 2(-1+i)e^{(-1+i)t}$

$$\int\limits_c (\bar{z})^2\, dz, = \int_0^{\frac{\pi}{4}} (2e^{(-1-i)t})^2 \cdot 2(-1+i)e^{(-1+i)t}\, dt = 8(-1+i)\int_0^{\frac{\pi}{4}} e^{(-3-i)t}\, dt$$

$$= 8\frac{-1+i}{-3-i}\left(e^{(-3-i)\frac{\pi}{4}} - e^0\right) = 8\frac{(-1+i)(-3+i)}{(-3-i)(-3+i)}\left(\frac{\sqrt{2}}{2}e^{-\frac{3\pi}{4}}(1-i) - 1\right)$$

$$= \frac{4}{5}(2-4i)\left(\frac{\sqrt{2}}{2}e^{-\frac{3\pi}{4}}(1-i) - 1\right)$$

d) The function is analytic in $\mathbb{C}$. The value of the integral only depends on starting and end point of the curve.

$$\int\limits_C e^{3z}\, dz = \left[\frac{e^{3z}}{3}\right]_0^{1+i\pi} = \frac{1}{3}(-e^3 - 1)\,.$$

Exercise 2:

a) In which area is the Möbius transform $T(z) = \dfrac{az+b}{cz+d}$ angle preserving?

b) Is it possible to map the area

$$M_1 := \{z \in \mathbb{C} : |z| > 1,\ \mathrm{Re}\,(z) > 0,\ \mathrm{Im}\,(z) > 0\}$$

onto the interior of a real triangle via Möbius transform? Here, a real triangle means a triangle whose corners are finite.

c) The mapping $f: \ z \to e^{\frac{i\pi}{4}}\bar{z}$ describes a rotary reflection. Obviously, it does not cause length distortions. The size of the angles is also preserved. f as a transformation $f: \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable. Where is f complex differentiable? How does the result compare to the theorem from page 75 from the lecture notes?

Theorem: If $w = f(z)$ is a conformal mapping and continuously differentiable as function $f: \mathbb{R}^2 \to \mathbb{R}^2$, it follows that $f(z)$ is complex differentiable and $f'(z) \neq 0$.

d) The area $G := \{z \in \mathbb{C} : z = re^{i\varphi},\ -\frac{\pi}{8} < \varphi < \frac{\pi}{8},\ 0 < r < 2\}$ is to be transformed onto the interior of the unit circle (bijective and conformal). Why does $z \mapsto \left(\dfrac{z}{2}\right)^8$ not do that?

Additional exercise: Give a bijective, conformal mapping that achieves this task.

Solution:

a) The Möbius transform is analytic in $\mathbb{C}$ without $z = -\dfrac{d}{c}$. With the exception of that point $T'(z) \neq 0$ holds everywhere. Hence, the given Möbius transform is angle preserving in all points except for $z = -\dfrac{d}{c}$.

b) Möbius transforms are angle preserving everywhere except for $z = -\dfrac{d}{c}$. Since a real triangle is to be constructed, no "corneröf the domain coincides with $z = -\dfrac{d}{c}$. The generalized circles (with boundary) intersect one another in the corners $1 + 0 \cdot i$ and $0 + i$ of M_1 with an angle of $\pi/2$. But both right angles cannot be reproduced inside a real triangle.

Alternatively: The boundary of the area is composed of parts of three generalized circles that have no common intersection. As a result, no Möbius transform can map all three generalized circles of the boundary onto lines since the lines of the image would intersect each other at infinity.

c) The function $g: \ z \to \bar{z}$ is not differentiable in $\mathbb{C}$ anywhere since

$$g(z) = x - iy \implies u_x = 1 \neq -1 = v_y.$$

Hence, f is not complex differentiable anywhere. Additionally, the mapping does not preserve angles. While it preserves the size of angles, it does not preserve the orientation.

d) $\left(\frac{z}{2}\right)^8$ only yields the sliced circular disk (along the negative real axis).

Solution for the additional exercise:

Step 1: $f_1(z) = \hat{z} = z^4$. *The given eighth of a circle is mapped bijectively and conformal onto a half circle. The boundary is defined by two generalized circles.*

Step 2: *The intersection of two generalized circles is mapped onto a sector if the intersections of the generalized circles (here:* $16i$ *and* $-16i$ *) are mapped onto* 0 *and* ∞. *We choose* $\tilde{z} = f_2(\hat{z}) := \dfrac{16i + \hat{z}}{16i - \hat{z}}$. *The image of* $i\mathbb{R}$ *is* $\mathbb{R}$ *(form of the coefficients!) and the image of the circle is a line that is perpendicular to* $\mathbb{R}$ *in* $T(-16i) = 0$ *(angle preserving). The circular boundary is mapped onto the imaginary axis. The right half of the circular disk is mapped onto the fourth quadrant since* $T(0) = 1$ *and* $T(16) = -i$.

Step 3: $W = f_3(\tilde{z}) = \tilde{z}^2$. *We double the opening angle and transform the boundary onto a line (the real axis).*

Step 4: *We map the real axis onto the unit circle in such a way that* $-i$ *is mapped onto the origin. By doing this, we achieve that the lower half plane is mapped onto the interior of the unit circle. The transform* $w = f_4(W) := \dfrac{W+i}{W-i}$ *achieves this.*

Finally we get

$$f(z) = \frac{\left(\dfrac{16i + z^4}{16i - z^4}\right)^2 + i}{\left(\dfrac{16i + z^4}{16i - z^4}\right)^2 - i}.$$

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