Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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Complex functions for Engineering Students

Sheet 5 (Homework)

Exercise 1: Given is a hollow, very long circular cylinder with radius one. Let the top and bottom half be electrically isolated from one another. The upper half has the electric potential $\Phi = 100$ V while the lower one has a potential of $\Phi = -100$ V. With an adequate choice in coordinate systems the intersection of the cylinder with the complex plane yields:

$$\rho^2 \cdot \frac{\partial^2}{\partial \rho^2} \Psi + \rho \cdot \frac{\partial}{\partial \rho} \Psi + \frac{\partial^2}{\partial \alpha^2} \Psi = 0$$

Calculate the potential and field strength inside the cylinder.

Hint : Transform the unit circle onto a sector (e.g. the right half plane). With an adequate transformation, the boundary values for the model are only dependent on the angle. Use the potential equation for polar representation to solve the differential equation in the model domain

$$r^2 \cdot \frac{\partial^2}{\partial r^2} \Psi + r \cdot \frac{\partial}{\partial r} \Psi + \frac{\partial^2}{\partial \phi^2} \Psi = 0$$

while taking the special structure of the boundary values into account. Rewrite the solution for the model domain into cartesian representation and use an inverse transform.

Solution:

The two boundary parts have two intersections:

$\longrightarrow$ create a sector

Circle: transform onto a line via Möbius transform.

Sector is bounded by the line: e.g. $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$.

Transform -1 and 1 onto zero and ∞.

Choose real coefficients: $\mathbb{R}$ remains $\mathbb{R}$.

The Möbius transform $T(z) := \dfrac{1+z}{1-z}$ maps the interior of the unit cirlce onto the right half space since

$$T(-1) = 0, \quad T(1) = \infty, \quad T(i) = i \quad T(0) = 1\,.$$

In the model domain the problem is defined as follows:

$$\begin{aligned} \Delta\Psi(u,v) &= 0\,, &\quad \text{for } u>0\,,\\ \Delta\Psi(0,v) &= 100\,, &\quad \text{for } v>0\,,\\ \Delta\Psi(0,v) &= -100\,, &\quad \text{for } v<0\,. \end{aligned}$$

Not only do we transform into polar coordinates but due to the structure of the boundary values we make the ansatz $\Psi = \Psi(\alpha)$. The resulting IBVP is

$$\frac{\partial^2}{\partial\alpha^2}\Psi = 0 \qquad \Psi(\pi/2) = 100, \qquad \Psi(-\pi/2) = -100$$

with the solution

$$\Psi(\rho,\alpha) = \frac{200}{\pi}\,\alpha, \qquad \text{or} \qquad \Psi(u,v) = \frac{200}{\pi}\arctan\frac{v}{u}$$

Hence, in the physical domain we obtain

$$\Phi(z) = \frac{200}{\pi}\arctan\frac{\mathrm{Im}\,(T(z))}{\mathrm{Re}\,(T(z))} = \frac{200}{\pi}\arctan\frac{2y}{1-x^2-y^2}$$

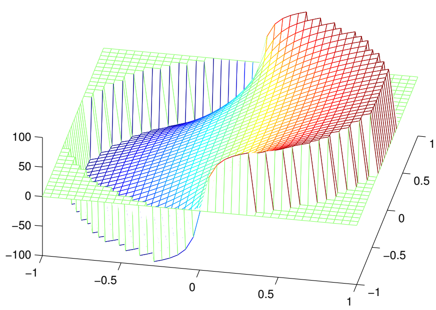
where the last equation is obtained by using $T(z)$ and fundamental simplifications.

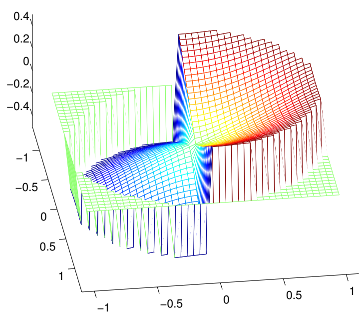
For the field strength it holds that

$E = -\,\mathrm{grad}(\Phi) = -\Phi_x - i\Phi_y = -\,\mathrm{grad}(\Psi(T(z))\overline{T'(z)}$. So

$$E(z) = -\,\mathrm{grad}\,\Psi\left(\frac{1+z}{1-z}\right)\overline{\left(\frac{2}{(1-z)^2}\right)}$$

or by explicit derivation of $\Phi(x,y) = \dfrac{200}{\pi}\arctan\dfrac{2y}{1-x^2-y^2}$:





$$\begin{aligned} E &= -\,\mathrm{grad}(\Phi) = -\Phi_x - i\Phi_y\\ &= -\frac{200}{\pi((1-x^2-y^2)^2+4y^2)}\left(4xy + i(2-2x^2+2y^2)\right) \end{aligned}$$

Supplement (not expected from students):

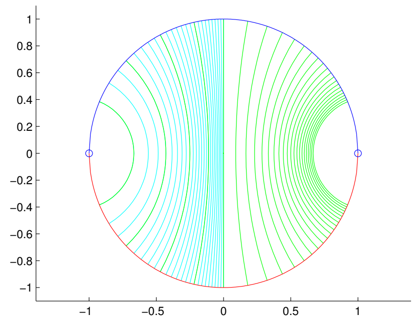
With the velocity potential $\tilde{U} = \Psi$ inside the model domain one can obtain:

$$\nabla \tilde{U} = \frac{200}{\pi} \begin{pmatrix} \dfrac{-v}{u^2+v^2} \\ \dfrac{u}{u^2+v^2} \end{pmatrix}$$

Hence, for the stream function it follows that $\nabla \tilde{V} = \dfrac{200}{\pi} \begin{pmatrix} \dfrac{u}{u^2+v^2} \\ \dfrac{v}{u^2+v^2} \end{pmatrix}$

and as a result $\tilde{V}(u,v) = \dfrac{100}{\pi} \ln\left(u^2+v^2\right)$

The streamlines can be obtained via inverse transform of $\tilde{V}$ with $f^{-1}(w) = \dfrac{w-1}{w+1}$ and plotting of the respective contour lines.



Exercise 2:

a) For $z = x + iy$ let $\bar{z} = x - iy$. Calculate

$$\oint_c \bar{z} \cdot z^{\frac{1}{2}}\, dz$$

along the curve

$$c : [0, \pi] \to \mathbb{C},\ c(t) = 4e^{it} \text{ und } C : [0, \pi] \to \mathbb{C},\ C(t) = 4e^{-it}$$

and confirm that the complex line integral is path-dependent in general.

b) Determine the values of the following line integrals if they exist. The curves are to be traversed once in positive direction.

i) $\int_{C_1} \frac{1}{z-2}\, dz$ C_1 : Circle with radius 1 around the origin,

ii) $\int_{C_2} \frac{1}{z-2}\, dz$ C_2 : Circle with radius 2 around the origin,

iii) $\int_{C_3} \frac{1}{z-2}\, dz$ C_3 : Circle with radius 3 around the origin.

Solution for exercise 2:

a) $f(c(t)) = 4e^{-it} \cdot (4e^{it})^{\frac{1}{2}} = 4e^{-it} \cdot 2e^{\frac{it}{2}} = 8e^{-\frac{it}{2}}$,

$\dot{c}(t) = 4ie^{it}$, $\quad f(c(t))\dot{c}(t) = 4ie^{it} 8e^{-\frac{it}{2}} = 32ie^{\frac{it}{2}}$.

$$\int_c \bar{z}\, z^{\frac{1}{2}}\, dz, = \int_0^\pi 32ie^{\frac{it}{2}}\, dt$$

$$= 32i \left[\frac{e^{\frac{it}{2}}}{\frac{i}{2}} \right]_0^\pi = 64(e^{i\frac{\pi}{2}} - e^0) = 64(i-1)\,.$$

$f(C(t)) = 4e^{it} \cdot (4e^{-it})^{\frac{1}{2}} = 4e^{it} \cdot 2e^{\frac{-it}{2}} = 8e^{\frac{it}{2}}$,

$\dot{C}(t) = -4ie^{-it}$, $\quad f(C(t))\dot{C}(t) = -4ie^{-it} 8e^{\frac{it}{2}} = -32ie^{\frac{-it}{2}}$.

$$\int_C \bar{z}\, z^{\frac{1}{2}}\, dz, = \int_0^\pi -32ie^{\frac{-it}{2}}\, dt$$

$$= -32i \left[\frac{e^{\frac{-it}{2}}}{-\frac{i}{2}} \right]_0^\pi = 64(e^{i\frac{-\pi}{2}} - e^0) = 64(-i-1)\,.$$

b)

i) $\int_{C_1} \frac{1}{z-2}\, dz = 0$ Lecture page 97/CIT.

ii) $\int_{C_2} \frac{1}{z-2}\, dz$ does not exist since the singularity is located on the curve.

iii) $\int_{C_3} \frac{1}{z-2}\, dz = 2\pi i.$ Lecture page 94.

Exercise 3:

a) Given are the functions

$$f_1(z) := \frac{e^z - 1}{e^z + e^{-z}}\,, \qquad f_2(z) := \frac{1}{\ln(3-z)}\,, \qquad f_3(z) := \frac{1}{\ln(\frac{i}{2} - 4 - z)}.$$

Determine the radius of the largest circle around the origin for which the respective Taylor series T_k of f_k at zero converges to f_k (for $k = 1, 2, 3$). Do not explicitly calculate the series.

b) The function $f(z) = \dfrac{1}{z(z^2 - 4z + 13)}$ is to be approximated by a Taylor series at $z_0 := x_0 + iy_0$, $x_0 \in \mathbb{R}^+$, $y_0 \in \mathbb{R}$, that converges to $f(z)$ in the circular disk $|z - z_0| < |z_0|$ (at minimum). How must z_0 be chosen so that x_0 becomes as large as possible.

Solution for exercise 3:

a) f_1 : the denominator becomes zero for

$e^{2z} = e^{2x} \cdot e^{2yi} = -1 = e^{i\pi} \iff x = 0\,, \quad y = k\pi + \frac{\pi}{2}\,.$

The series converges to f inside the circle with radius $r = \dfrac{\pi}{2}\,.$

f_2 is not defined for $3 - z = 1$ or $3 - z = x \in (-\infty, 0\,.$

So $z = 2$ or $z = x \in [3, \infty)\,.$

The Taylor series for f_2 converges inside the circle with radius $r_2 = 2$ (around the origin).

f_3 is not defined for

$\frac{i}{2} - 4 - z = 1 \iff z = \frac{i}{2} - 5$

or

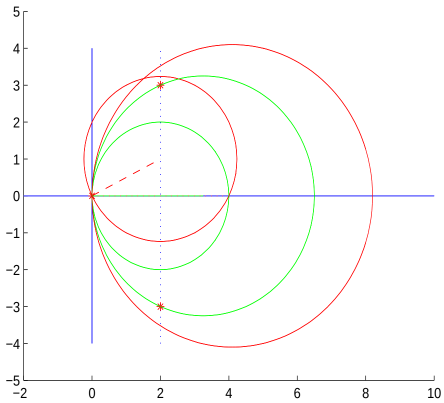
$\frac{i}{2} - 4 - z = 1 \in (-\infty, 0] \iff z = x + \frac{i}{2}$ with $x \in ([-4, \infty)\,.$

And neither for $z = \frac{i}{2}\,.$

The Taylor series for f_3 converges inside the circle with radius $r_3 = \frac{1}{2}$ (around the origin).

b) The denominator has the roots $z_1 = 0$, $z_{2,3} = 2 \pm 3i\,.$ z_0 has to be chosen in such a way that all roots lie on a circle around that point. Due to the symmetry of z_2, z_3 z_0 lies on the real axis. The following condition has to be satisfied

$$|z - 0| = |z - (2 \pm 3i)| \iff x_0^2 = (x_0 - 2)^2 + 9 \iff x_0 = \frac{13}{4} = z_0\,.$$



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