Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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Complex functions for Engineering Students

Solutions for work sheet 4

Exercise 1:

a) Determine a Möbius transform $T : \mathbb{C}^* \to \mathbb{C}^*, \quad T(z) := \dfrac{az+b}{cz+d}$ with

$$T(i) = 0\,, \qquad T(0) = 2\,, \qquad T(2i) = \infty\,.$$

b) Determine the images of the following generalized circles using T.

(i) $K :=$ imaginary axis,
(ii) $K_2 := \{\, z \in \mathbb{C} : |z| = 2 \,\}$,
(iii) $\tilde{K} :=$ real axis.

c) Determine the image of the quarter plane

$$S := \{\, z \in \mathbb{C} : \mathrm{Re}\,(z) > 0, \ \mathrm{Im}\,(z) > 0 \,\}\,.$$

d) Determine the image of

$$H := \{\, z \in \mathbb{C} : \mathrm{Re}\,(z) > 3 \,\}\,.$$

Solution for exercise 1)

a) $T(i) = 0\,, T(2i) = \infty \iff \quad T(z) = \dfrac{a(z-i)}{z-2i}.$

$T(0) = 2\,, \quad \Longrightarrow \quad T(z) = \dfrac{4(z-i)}{z-2i}.$

b) (i) Because of the given images of 0, i, $2i$ the transform is $T(i\mathbb{R}) = \mathbb{R}$.
Alternatively: $2i \in i\mathbb{R} \iff T(i\mathbb{R})$ is a line.
$T(0) = 2,\ T(\infty) = 4 \iff T(i\mathbb{R}) = \mathbb{R}$

(ii) $2i \in K_2 \iff T(K_2)$ is a line.
K_2 is symmetric to $i\mathbb{R} \implies T(K_2)$ is symmetric to $\mathbb{R}$.
$T(-2i) = \dfrac{4(-3i)}{-4i} = 3 \iff T(K_2)$ is the parallel line to the imaginary axis through the point 3

$$g_2 := T(K_2) = \{w \in \mathbb{C} : w = 3 + iv,\ v \in \mathbb{R}\}.$$

(iii) $2i \notin \mathbb{R} \Longleftrightarrow T(\mathbb{R})$ is a real circle K_R.

$\mathbb{R}$ is symmetric to $i\mathbb{R}$ and $K_2 \implies T(\mathbb{R})$ is symmetric to $\mathbb{R}$ and g_2.

The center of the circle in the image is $M = 3$.

Because of $T(0) = 2$ the radius $R = 1$.

c) The image of the quarter plane is bounded by $\mathbb{R}$ and K_R.

Now, we can calculate:

$T(2i) = \infty \implies$ upper halfspace $\longrightarrow$ outside of K_R.

$T(2) = \dfrac{8-4i}{2-2i} = \dfrac{2(2-i)(1+i)}{1-i^2} = 2 + 2i - i + 1 = 3 + i$.

The right halfspace is mapped onto the upper halfspace.

$$T(S) = \{w \in \mathbb{C} : \operatorname{Im}(w) > 0, \, |w-3| > 1\}.$$

Or for example:

$T(1+2i) = \dfrac{4+8i-4i}{1+2i-2i} = 4+4i$

Lies in the upper halfspace and outside of K_R, so

$$T(S) = \{w \in \mathbb{C} : \operatorname{Im}(w) > 0, \, |w-3| > 1\}.$$

d) The image of the line $\operatorname{Re}(z) = 3$ is a real circle, since $2i$ does not lie on the line. The center C and ∞ are symmetric to the circle in the image. Hence

$T^{-1}(\infty) = 2i$ is symmetric to $T^{-1}(C)$ regarding the line $\operatorname{Re}(z) = 3$.

$\implies T^{-1}(C) = 2i + 6 \Longleftrightarrow C = T(2i+6) = \dfrac{4(2i+6-i)}{2i+6-2i} = 4 + \frac{2i}{3}$

Because of $T(\infty) = 4$ the circle's radius is $r = \frac{2}{3}$. Because of $T(0) = 2$ $E : \operatorname{Re}(z) > 3$ is mapped onto the interior of the circle.

$$T(H) = \{w \in \mathbb{C} : |w - 4 - \frac{2i}{3}| < \frac{2}{3}\}.$$

Exercise 2:

In which points of their domain are the following functions complex differentiable?

a) $f_1 : \mathbb{C} \to \mathbb{C},\ f_1(z) = \mathrm{Re}\,(z) \cdot \mathrm{Im}\,(z)\,.$

b) $f_2 : \mathbb{C} \to \mathbb{C},$
$f_2(z) = (\mathrm{Re}\,(z) + 2)^2 - (\mathrm{Im}\,(z) + 2)^2 + i\Big[\mathrm{Im}\,(z)(\mathrm{Re}\,(z) + 4) + \mathrm{Re}\,(z)(\mathrm{Im}\,(z) + 4)\Big]\cdot$

c) $f_3 : \mathbb{C} \setminus \{0\} \to \mathbb{C},\ f_3(z) = \dfrac{z^2}{\bar{z}}\,.$
Hint: Use Cauchy Riemann equations in polar representation: $u_r = \frac{1}{r} v_\varphi$ and $v_r = -\frac{1}{r} u_\varphi\,.$

Solution:

a) $f_1(z) = \mathrm{Re}\,(z) \cdot \mathrm{Im}\,(z) = xy + 0 \cdot i \iff u = xy,\ v = 0$
We verify the Cauchy Riemann equations:
$u_x = v_y \iff y = 0, \qquad \text{and} \qquad u_y = -v_x \iff x = 0$
The function is only complex differentiable in zero.

b) $f_2(z) = (\mathrm{Re}\,(z) + 2)^2 - (\mathrm{Im}\,(z) + 2)^2 + i\Big[\mathrm{Im}\,(z)(\mathrm{Re}\,(z) + 4) + \mathrm{Re}\,(z)(\mathrm{Im}\,(z) + 4)\Big]\cdot$
$u = (x + 2)^2 - (y + 2)^2$ and $v = y(x + 4) + x(y + 4)$
So $u_x = 2(x + 2) = v_y = x + 4 + x$ and $-u_y = 2(y + 2) = v_x = y + y + 4\,.$
The function is complex differentiable everywhere. It holds that $f_2(z) = (z + 2 + 2i)^2 - 8i\,.$

c) $f_3(z) = \dfrac{z^2}{\bar{z}} = \dfrac{r^2 e^{2i\varphi}}{r e^{-i\varphi}} = r e^{3i\varphi} = r \cdot \cos(3\varphi) + i \cdot r \cdot \sin(3\varphi)$
$u_r = \cos(3\varphi), \qquad \frac{1}{r} v_\varphi = 3 \cdot \cos(3\varphi)$
f_3 is only complex differentiable in points with $\cos(3\phi) = 0\,.$
$v_r = \sin(3\varphi), \qquad \frac{1}{r} u_\varphi = -3 \cdot \sin(3\varphi)$
f_3 is only complex differentiable in points with $\sin(3\phi) = 0\,.$
There is no angle for which both sine and cosine are zero. Hence, f_3 is complex differentiable nowhere.

Alternatively:

The function $f_4(z) = \bar{z} = x - iy$ ist not holomorphic because $u_x = 1 \neq v_y = -1$. If f_3 were holomorphic, so too would

$z^2 \cdot (f_3(z))^{-1}$ be holomorphic!

Exercise 3:

Hint : You do not need to give exact transformations.

a) To solve a potential problem, the area outside the two circles
$K_1 := \left\{z \in \mathbb{C} \,:\, |z-2| \,\leq\, \frac{3}{2}\right\}$, and
$K_2 := \left\{z \in \mathbb{C} \,:\, |z+1| \,\leq\, \frac{3}{2}\right\}$
is to be mapped onto a strip parallel to or onto the interior of the circle around the origin. Which of these two transformations is achievable by the use of a Möbius transform?

b) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ be fixed parameters. Which of the following areas can be mapped onto a sector of the form

$$S := \left\{ w \in \mathbb{C} \,:\, w = r\, e^{i\phi},\, r \in \mathbb{R}^+,\, -\pi \,<\varphi_1 \,<\, \phi \,<\, \varphi_2 \,<\, \pi \right\}$$

using Möbius transforms? Please explain your answer.

(i)
$$G_1 := \{\, z \in \mathbb{C} \,:\, \alpha < |z| < \beta \} \,.$$

(ii)
$$G_2 := \{\, z \in \mathbb{C} \,:\, \alpha < \mathrm{Re}\,(z) \,<\, \beta \} \,.$$

(iii)
$$G_3 := \left\{ z \in \mathbb{C} \,:\, |z - \alpha| < \frac{3}{4}\,|\beta - \alpha|\,,\, |z - \beta| < \frac{3}{4}\,|\beta - \alpha| \right\} .$$

Solution for exercise 3)

a) The domain is bounded by the two circles
$C_1 := \left\{z \in \mathbb{C} \,:\, |z-2| \,=\, \frac{3}{2}\right\}$, and
$C_2 := \left\{z \in \mathbb{C} \,:\, |z+1| \,=\, \frac{3}{2}\right\}$.
These two circles intersect at $z_s = \frac{1}{2}$. To map them onto a ring, the circles would have to be mapped onto two circles that do not intersect each other which is not possible!

For mapping to a strip, the circles have to be mapped onto two lines g_1 und g_2. These have an intersection at ∞. The intersection of both circles is mapped to the point at ∞ and we obtain two lines as images of the circles C_1 and C_2. Since the circles have no other intersections the images (lines) also have no other intersections. Hence, they are parallel to each another.
If real coefficients are chosen, the centers M_1 and M_2 of the circles C_1 and C_2 are mapped onto real numbers, g_1 and g_2 are perpendicular to $\mathbb{R}$ and point at ∞ is mapped onto $(T(M_1) + T(M_2))/2$. The area outside of both cicles is mapped onto the parallel strip between g_1 and g_2.

Hence, a mapping onto a strip is possible. For example $T(z) = \dfrac{1}{z - \frac{1}{2}}$.

b) A sector S of the given form is bounded by two generalized circles with the intersections zero and ∞.

(i) The circular area

$$G_1 := \{ z \in \mathbb{C} : \alpha < |z| < \beta \} .$$

is bounded by two generalized circles without intersection. The mapping from G_1 onto S using a Möbius transform is not possible.

(ii) The parallel strip

$$G_2 := \{ z \in \mathbb{C} : \alpha < \operatorname{Re}(z) < \beta \} .$$

is bounded by two generalized circles with an intersection at ∞. The mapping from G_2 onto S using a Möbius transform is not possible.

(iii) The area

$$G_3 := \left\{ z \in \mathbb{C} : |z - \alpha| < \frac{3}{4}|\beta - \alpha| , |z - \beta| < \frac{3}{4}|\beta - \alpha| \right\} .$$

is bounded by two generalized circles with intersections at z_1 and z_2. If one of the intersections is mapped onto zero and the other onto ∞, a Möbius transform maps G_3 to a sector.

Class: 27.05.24