Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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Complex functions for Engineering Students

Solutions for sheet 4 (Homework)

Exercise 1:

a) Give a Möbius transform that satisfies

$$T(0) = 2i,\ T(4) = 0,\ T(8) = \infty\,.$$

b) (i) Determine the images of the following lines while using the mapping T from part a). Explain your results.

A) $g_1 = \{z \in \mathbb{C}^* : \operatorname{Im}(z) = 0\}$.

B) $g_2 = \{z \in \mathbb{C}^* : \operatorname{Im}(z) = 8 - \operatorname{Re}(z)\}$.

C) $g_3 = \{z \in \mathbb{C}^* : \operatorname{Re}(z) = \operatorname{Im}(z)\}$.

(ii) Onto which set is the interior of the triangle with the corners $0, 8, 4+4i$ mapped? Sketch image and domain in the complex plane!

Solution for exercise 1:)

a) $T(4) = 0,\ T(8) = \infty \Longrightarrow T(z) = a \cdot \dfrac{z-4}{z-8}$.

$T(0) = \frac{a}{2} = 2i \Longrightarrow T(z) = 4i \cdot \dfrac{z-4}{z-8}$.

b) (i) Generalized circles through 8 are mapped onto lines.

A) The image of the real axis is a line which satisfies

$$T(0) = 2i, \quad T(4) = 0.$$

Hence $T(\mathbb{R}) = i \cdot \mathbb{R}$.

B) The image of $g_2 = \{z \in \mathbb{C}^* : \operatorname{Im}(z) = 8 - \operatorname{Re}(z)\}$ is a line as well since $8 \in g_2$. It holds that $T(\infty) = 4i$ and for example

$T(8i) = 4i \cdot \dfrac{8i-4}{8i-8} = 2i \cdot \dfrac{2i-1}{i-1} = 2i \cdot \dfrac{(2i-1)(-i-1)}{-i^2+1^2} = 1+3i$

or

$T(4+4i) = 4i \cdot \dfrac{4i}{4i-4} = -4 \cdot \dfrac{-i-1}{-i^2+1^2} = 2+2i$.

$T(g_2) = \{z \in \mathbb{C}^* : \operatorname{Im}(z) = 4 - \operatorname{Re}(z)\}$.

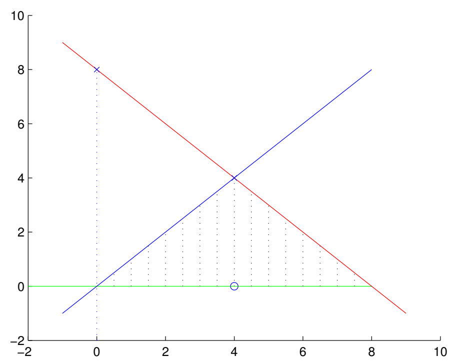
C) The image of g_3 is a real circle K since $8 \notin g_3$.
The circle in the image goes through the points
$T(4+4i) = 2+2i, T(0) = 2i, T(\infty) = 4i$
The center of that circle lies on the bisector of the connecting line between $2+2i$ and $2i$. So $M = 1+ib$.
Addtionally, the center also lies on the bisector of the connecting line between $4i$ and $2i$. As a result, $M = 1+3i$.
The radius is obtained by
$R = \sqrt{(1-0)^2+(3-2)^2} = \sqrt{2}$ since $2i \in$ circle in the image.

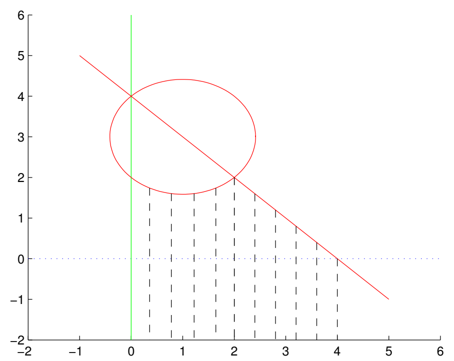
(ii) The triangle is bounded by the lines g_1, g_2, g_3. Hence, the image of the triangle is bounded by the images of these lines.

The image is located outside of K since $T(4) = 0$ and below of $T(g_2)$. Since $T(4+4i) = 2+2i$ the image is located on the right side of $T(g_1)$ because of $T(4+i) = \frac{8}{5}(1+i)$.

Alternatively, the image of a point from the interior of the triangle can be calculated. The image that is obtained is given by the points $w = u+iv \in \mathbb{C}$ with:

$u > 0,\ v < 4-u,\ |w-(1+3i)| > \sqrt{2}$.





Exercise 2:

Let i be the imaginary unit and $z = x + iy,\ x, y \in \mathbb{R}$.

a) For which $k, l \in \mathbb{R}$ is the function

$$f : \mathbb{C} \to \mathbb{C}\,,\ f(z) := (x^3 + kxy^2) + i \cdot \left(lx^2y - y^3\right)$$

complex differentiable in every point $\in \mathbb{C}$?

b) Given the function

$$u(x + iy) = \mathrm{Re}\,(f(x + iy)) = 3\cos(4x)e^{4y}\,.$$

i) Show that the function u is harmonic.

ii) Determine all conjugated harmonic functions v to u, so that all functions v, for which $f = u + iv$, are complex differentiable everywhere in $\mathbb{C}$.

Solution:

a) Using the usual notation $z = x + iy$:

$$f(z) = \underbrace{(x^3 + kxy^2)}_{u(x,y)} + i \cdot \underbrace{\left(lx^2y - y^3\right)}_{v(x,y)}.$$

The Cauchy Riemann equations yield:

$$u_x = 3x^2 + ky^2 \overset{!}{=} v_y = lx^2 - 3y^2 \text{ so } \boxed{-k = l = 3}$$

and

$$-u_y = -2kxy \overset{!}{=} v_x = 2lxy \text{ so } \boxed{k = -l}.$$

For $k = -3$ and $l = 3$ f is complex differentiable for all of $\mathbb{C}$.

b) i) $u_{xx} = \left(-12\sin(4x)e^{4y}\right)_x = -48\cos(4x)e^{4y}$.

$$u_{yy} = \left(+12\cos(4x)e^{4y}\right)_y = 48\cos(4x)e^{4y}.$$

So $\Delta u = u_{xx} + u_{yy} = 0$.

ii) $f(z) = u(z) + iv(z),\ u(x + iy) = \mathrm{Re}\,(f(x + iy)) = 3\cos(4x)e^{4y}$.

$$v_y = u_x = -12\sin(4x)e^{4y} \iff v(x,y) = -3\sin(4x)e^{4y} + c(x)\,,$$

$$-u_y = -12\cos(4x)e^{4y} \overset{!}{=} v_x = -12\cos(4x)e^{4y} + c'(x)$$

$$\iff c'(x) = 0 \implies v(x,y) = -3\sin(4x)e^{4y} + C\,, \qquad C \in \mathbb{R}.$$

Exercise 3:

To solve a potential problem, the area outside the two circles

$\tilde{K}_1 := \left\{ z \in \mathbb{C} \,:\, |z - \tfrac{5}{2}| \le \tfrac{3}{2} \right\}$, and

$\tilde{K}_2 := \left\{ z \in \mathbb{C} \,:\, |z + \tfrac{5}{2}| \le \tfrac{3}{2} \right\}$

is to be mapped onto the interior of a circular disk around the origin. Present an adequate mapping.

Solution:

The lecture slides (pages 60 and 61) can be used or one can derive the following results by oneself. Let K_1 und K_2 be the boundaries of $\tilde{K}_1$ und $\tilde{K}_2$. We use a Möbius transform since it maps both circles onto concentric circle for the image.

For the image, zero and the point at infinity are symmetric to both circles of the image. Hence, these have to be the images of the two points p_1, p_2 that are symmetric to both circles K_1 and K_2 regarding the domain. The points p_1 and p_2 lie on the connecting line between the centers of K_1 and K_2 (so on the real axis).

Because of their symmetry of both circles to the imaginary axis, it holds that $p_1 = -p_2 =: p$. The condition for symmetry regarding K_2 now yields:

$$(p - \frac{5}{2}) \cdot (-p - \frac{5}{2}) = \left(\frac{3}{2}\right)^2 \Leftrightarrow p^2 = \left(\frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = 4\,.$$

We choose $p_1 = -2$ and $p_2 = 2$.

For $T(z) := \dfrac{z-2}{z+2}$ it holds that

- The real axis is mapped onto the real axis.
- The circles K_1 and K_2 are mapped onto real cirlces that are symmetric to the image of the real axis. The center points of the circles of the images lie on $\mathbb{R} = T(\mathbb{R})$.
- It holds that:
  $T(-4) := \dfrac{-4-2}{-4+2} = 3, \qquad T(-1) := \dfrac{-1-2}{-1+2} = -3\,.$
  The image of K_1 is the circle with radius 3 around 0.

  $T(4) := \dfrac{4-2}{4+2} = \frac{1}{3}, \qquad T(1) := \dfrac{1-2}{1+2} = -\frac{1}{3}\,.$
  The image of K_2 is the circle with radius $\frac{1}{3}$ around 0
  Because of $T(0) = -1$, the area between the circles is mapped onto the following circular section:
  $$R := \left\{ z \in \mathbb{C} \,:\, \frac{1}{3} < |z| < 3 \right\}\,.$$

Hand in: 27.05-31.05.24