**Fachbereich Mathematik der Universität Hamburg** **SoSe 2024**
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# Complex functions for Engineering Students

## Solutions for work sheet 3

### Exercise 1:

Let $\ln(z)$ be the principal value for the complex logarithm. Given the following complex numbers

$z_1 = \sqrt{2}(-1+i), \quad z_2 = 3i, \quad z_3 = -4i\,.$

a) Calculate the cartesian representations for

$z_4 := z_1 \cdot z_2, \quad z_5 := \frac{z_1}{z_2}, \quad z_6 := z_1 \cdot z_3, \quad z_7 := \frac{z_1}{z_3}.$

b) Calculate $\ln(z_k), \quad k = 1, 2, ..., 7\,.$

c) Compare

$\ln(z_1 \cdot z_k)$ with $\ln(z_1) + \ln(z_k)$
and
$\ln(\frac{z_1}{z_k})$ with $\ln(z_1) - \ln(z_k)$ for $k = 2, 3\,.$

d) For what complex numbers do the rules for $\mathbb{R}$ apply:

$\ln(a \cdot b) = \ln(a) + \ln(b), \qquad \ln(\frac{a}{b}) = \ln(a) - \ln(b)\,?$

### Solution:

a) Calculate the cartesian representations for

$z_4 := z_1 \cdot z_2 = \sqrt{2}(-1+i) \cdot 3i = 3\sqrt{2}(-1-i)\,,$

$z_5 := \frac{z_1}{z_2} = \frac{\sqrt{2}(-1+i)}{3i} = \frac{\sqrt{2}}{3}(1+i)\,,$

$z_6 := z_1 \cdot z_3 = \sqrt{2}(-1+i) \cdot (-4i) = 4\sqrt{2}(1+i)\,,$

$z_7 := \frac{z_1}{z_3} = \frac{\sqrt{2}(-1+i)}{-4i} = \frac{\sqrt{2}}{4}(-1-i)\,,$

b) It holds that $\ln(z) = \ln(|z|) + i\arg(z)$ while $\arg(z) \in (-\pi, \pi)$. One obtains:

$z_1 = 2e^{i\frac{3\pi}{4}} \implies \ln(z_1) = \ln(2) + i\frac{3\pi}{4}$

$z_2 = 3e^{i\frac{\pi}{2}} \implies \ln(z_2) = \ln(3) + i\frac{\pi}{2}$

$z_3 = 4e^{-i\frac{\pi}{2}} \implies \ln(z_3) = \ln(4) - i\frac{\pi}{2}\,.$

$z_4 = z_1 \cdot z_2 = 6e^{i\frac{5\pi}{4}} = 6e^{-i\frac{3\pi}{4}} \implies \ln(z_4) = \ln(6) - i\frac{3\pi}{4} \quad ,$

$z_5 := \frac{z_1}{z_2} = \frac{2}{3}e^{i\frac{\pi}{4}} \implies \ln(z_5) = \ln(\frac{2}{3}) + i\frac{\pi}{4}$ ,

$z_6 := z_1 \cdot z_3 = 8e^{i\frac{\pi}{4}} \implies \ln(z_6) = \ln(8) + i\frac{\pi}{4}$ ,

$z_7 := \frac{z_1}{z_3} = \frac{1}{2}e^{i\frac{5\pi}{4}} = \frac{1}{2}e^{-i\frac{3\pi}{4}} \implies \ln(z_7) = \ln(\frac{1}{2}) - i\frac{3\pi}{4}$ .

c) $\ln(z_1 \cdot z_2) = \ln(6) - i\frac{3\pi}{4} \neq \ln(z_1) + \ln(z_2) = \ln(6) + i\frac{5\pi}{4}$ .

$\ln(z_1 \cdot z_3) = \ln(8) + i\frac{\pi}{4} = \ln(z_1) + \ln(z_3)$ .

$\ln(\frac{z_1}{z_2}) = \ln(\frac{2}{3}) + i\frac{\pi}{4} = \ln(z_1) - \ln(z_2)$ .

$\ln(\frac{z_1}{z_3}) = \ln(\frac{1}{2}) - i\frac{3\pi}{4} \neq \ln(z_1) - \ln(z_3) = \ln(\frac{1}{2}) + i\frac{5\pi}{4}$ .

d) $\ln(c_1 \cdot c_2) = \ln(c_1) + \ln(c_2)$ is valid in $\mathbb{C}$ exactly when

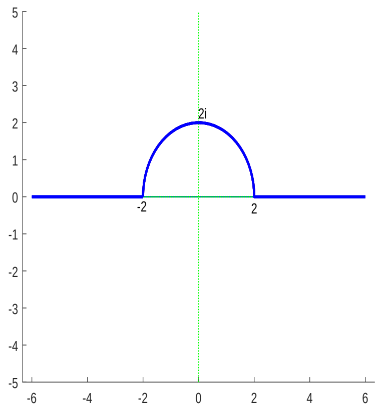
$-\pi < \arg(c_1) + \arg(c_2) < \pi$ .

$\ln(\frac{c_1}{c_2}) = \ln(c_1) - \ln(c_2)$ is valid in $\mathbb{C}$ exactly when

$-\pi < \arg(c_1) - \arg(c_2) < \pi$ .

**Exercise 2:** Let the two sets $D_1$ and $D_2$ be

$$D_1 := \{x \in \mathbb{R} : -\infty < x \leq -2\} \cup \left\{z \in \mathbb{C} : z = 2e^{i\phi},\, \phi \in ]0,\pi[\right\} \cup \{x \in \mathbb{R} : 2 \leq x < \infty\},$$




and

$$D_2 := \{x \in \mathbb{R} : -\infty < x \leq -2\} \cup \left\{z \in \mathbb{C} : z = 2e^{i\phi},\, \phi \in ]\pi,2\pi[\right\} \cup \{x \in \mathbb{R} : 2 \leq x < \infty\} .$$

Determine the images of $D_1$ and $D_2$ for the mapping $f(z) = \dfrac{2}{z} + \dfrac{z}{2}$.

On which of these sets $D_1$, $D_2$, $D_1 \cup D_2$ is $f$ invertible?

**Solution for exercise 2:**

$f$ is differentiable in $\mathbb{C} \setminus 0$ with $f'(z) = -\frac{2}{z^2} + \frac{1}{2}$.

It holds that $f'(z) = 0 \iff z = \pm 2$.

Hence, $f$ is monotone on the intervalls $]-\infty, -2]$ and $[2, \infty[$.

$$\lim_{z\to-\infty} f(z) = -\infty \text{ and } f(-2) = -2 \implies f(]-\infty,-2]) = ]-\infty,-2]$$

$$\lim_{z\to\infty} f(z) = \infty \text{ and } f(2) = 2 \implies f([2,\infty[) = [2,\infty[$$

$$f(2e^{i\phi}) = \frac{2}{2e^{i\phi}} + \frac{2e^{i\phi}}{2} = e^{-i\phi} + e^{i\phi} = 2\cos(\phi)$$

For $\phi \in ]0,\pi[$ $f(2e^{i\phi})$ passes through the intervall $]-2,2[$ in mathematically positive manner.

For $\phi \in ]\pi,2\pi[$ $f(2e^{i\phi})$ passes through the intervall $]-2,2[$ in mathematically negative manner.

Hence $f(D_1) = f(D_2) = \mathbb{R}$.

$f$ is invertible in $D_1$ and $D_2$ but not in $D_1 \cup D_2$.

**Exercise 3:**

a) (i) How many solutions does the equation $(z-2i)^{10} = z^{10}$ have?

(ii) Show that all solutions for the equation from i) lie on the line $\operatorname{Im}(z) = 1$.

b) How many solutions does the equation $(z-2i)^i = z^i$ have?

**Solution:**

a) (i) After subtracting $z^{10}$ from both sides the roots of a polynomial of degree nine in $\mathbb{C}$ need to be found. As a result, there are nine solutions (considerung possible multiplicity).

(ii)
$$(z-2i)^{10} = z^{10} \Longrightarrow |(z-2i)^{10}| = |z^0| \Longrightarrow |z-2i|^{10} = |z|^{10}$$
which means that $z$ has the same distance from 2i and 0. $z$ is located on the bisector of the line between 0 and 2i. Hence, the solutions lie on the line
$$z = x + i, \qquad x \in \mathbb{R}.$$

b) Solutions of $(z-2i)^i = z^i$:

Let $w := f(z) := z^i$. Hence
$$w = z^i = \exp(\ln(z))^i = \exp(\ln(z)\cdot i) = \exp\Big(i \cdot \ln(|z|) + i^2 \arg(z)\Big)$$
So
$$|w| = e^{-\arg(z)}, \qquad \arg(w) = \ln(|z|) + 2k\pi \qquad \text{for adequate } k \in \mathbb{Z}.$$
If $\tilde{w} := (z-2i)^i = z^i = w$ holds, we obtain
$|\tilde{w}| = |w|$. So
$$e^{-\arg(z)} = e^{-\arg(z-2i)} \Longrightarrow \arg(z) = \arg(z-2i) \Longrightarrow z = iy, y \in \mathbb{R}\setminus[0,2]$$
Furthermore,
$$\exp(i \cdot \ln(|z|)) = \exp(i \cdot \ln(|z-2i|))$$
has to hold. With $k \in \mathbb{Z}$:
$$\ln(|z-2i|) = \ln(|z|) + 2k\pi \underset{z=iy}{\Longrightarrow} \ln\left(\left|\frac{iy-2i}{iy}\right|\right) = 2k\pi \Longrightarrow \left|1 - \frac{2}{y}\right| = e^{2k\pi}$$
For $y < 0$ or $y > 2$ the absolute value is not needed:
$y_k = \dfrac{2}{1-e^{2k\pi}}$ with $k \in \mathbb{Z}, k \neq 0$.

There are infinitely many solutions
$$z_k = \frac{2i}{1 \pm e^{2k\pi}} \qquad k \in \mathbb{N}$$