Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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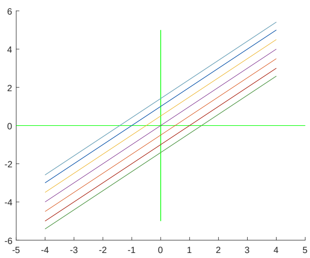
Complex functions for Engineering Students

Solutions for sheet 3 (Homework)

Exercise 1:

Give a function that maps the strip
$S := \left\{ z \in \mathbb{C} \,:\, \mathrm{Re}\,(z) - \sqrt{2} \,<\, \mathrm{Im}\,(z) \,<\, \mathrm{Re}\,(z) + \sqrt{2} \right\}$



to the circular ring $R := \{z \in \mathbb{C} : 1 < |z| < 2\}$. The function is supposed to use z directly and not use its imaginary or real part.

Hint: Transform S to a strip parralel to an axis $\tilde{S}$ first.

Solution:

The strip is bounded by two lines:

$g_1 : \mathrm{Im}\,(z) = \mathrm{Re}\,(z) - \sqrt{2}$

$g_2 : \mathrm{Im}\,(z) = \mathrm{Re}\,(z) + \sqrt{2}$

We can map strips that are parallel to the imaginary axis onto rings. Hence, we rotate the strip first:

Step 1 : Rotate by $\pi/4$

$f_1(z) = e^{i\frac{\pi}{4}} \cdot z \,=\, \frac{1}{\sqrt{2}}(1+i)z\,.$

The points $P_1 = 0 - i\sqrt{2},\ P_2 = \sqrt{2}$ lie on g_1 and are mapped onto $1 \pm i$. It follows that $f_1(g_1) = g_3 = \{\, z \in \mathbb{C} \,:\, \mathrm{Re}\,(z) = 1\}$.

Similarly, we obtains: $f_1(g_2) = g_4 = \{\, z \in \mathbb{C} \,:\, \mathrm{Re}\,(z) = -1\}$.

$f_1(S) = \{w = u + iv \,:\, -1 < u < 1,\ -\infty < v < \infty\} = \tilde{S}$

If the exponential function is applied to the image of the strip directly, we obtain:

$\exp(f_1(z)) = \exp(u + iv) = e^u \cdot e^{iv}$

$\exp(\tilde{S})$: Ring with inner radius e^{-1} , outer radius e^1

Goal: inner radius $1 = e^0$ and outer radius $= 2 = e^{\ln 2}$.

Step 2: We translate the strip

$f_2(z) := z + 1$

$f_2 \circ f_1(S) = \{w = u + iv \,:\, 0 < u < 2,\ -\infty < v < \infty\} = \hat{S}$

Step 3: We rescale

$f_3(z) := \frac{\ln(2)}{2} \cdot f_2 \circ\ f_1(z)$

$f_3(S) = \{w = u + iv \,:\, 0 < u < \ln(2),\ -\infty < v < \infty\}$

Step 4: Strip $\longrightarrow$ ring

$f_4(z) := \exp(f_3(z)) = \exp(u + iv) = e^u \cdot e^{iv}$

$f_4(S)$: Ring with inner radius 1, outer radius 2 around the origin.

Note: Suggestions of the form $f(z) = e^{i\mathrm{Im}\,(f_1(z))} \cdot \frac{(3+\mathrm{Re}\,(f_1(z)))}{2}$ yield the same result. If is more versed, one tends to avoid functions that include $\mathrm{Re}\, z,\ \mathrm{Im}\,(z), \bar{z}$ möglichst zu meiden. Usually, functions that are obtained in such a manner are not differentiable.

Exercise 2: Given the set $R = \{z \in \mathbb{C} : \frac{1}{4} < |z| < \frac{e^3}{4}, \operatorname{Re}(z) > 0\}$,

as well as the mapping

$$f(z) = e^{-i\frac{\pi}{2}} \cdot \ln(4z),$$

where ln is the principal value of the complex logarithm,

a) Sketch the set R in the complexen plane.

b) Determine the image of R obtained with the mapping f.

Solution for exercise 2:

a) Sketch: R is the right hakf if the ring around the origin with inner radius $\frac{1}{4}$ and outer radius $\frac{e^3}{4}$ without boundary. **[1 point]**

b) $f(z) = e^{-i\frac{\pi}{2}} \cdot \ln(4z)$.
Let $\tilde{w} = 4z$. Hence, $1 < |\tilde{w}| < e^3$ and $-\frac{\pi}{2} < \arg(\tilde{w}) < \frac{\pi}{2}$. **[1 point]**
For $\hat{w} = \ln(\tilde{w}) = \ln(|\tilde{w}|) + i \arg(\tilde{w})$ it holds that
$\operatorname{Re}(\hat{w}) \in]\ln(1), \ln(e^3)[=]0, 3[, \qquad \operatorname{Im}(\hat{w}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. **[1 point]**
Finally, we determine
$f(z) = e^{-i\frac{\pi}{2}} \cdot \hat{w}$ and obtain a rectangle parallel to the axes by rotation by $\frac{\pi}{2}$ (clockwise) with
$\operatorname{Re}(f(z)) \in (-\frac{\pi}{2}, \frac{\pi}{2}), \qquad \operatorname{Im}(f(z)) \in]-3, 0[$. **[1 point]**

Exercise 3) (4+3+3 Punkte)

a) For solving two potential problems, the following two transformations are to be executed:

(i) The boundary of the elliptical disk

$$E := \left\{ z = x + iy \in \mathbb{C} : \frac{16x^2}{25} + \frac{16y^2}{9} \le 1 \right\},$$

so $\mathbb{C} \backslash E$, is to be mapped to the outside of the unit circle $K_1 := \{w \in \mathbb{C} : |w| \le 1\}$.

(ii) The area between the two hyperbola branches given by $z = x + iy$ with

$$\frac{4x^2}{3} - 4y^2 = 1 \iff \frac{x^2}{\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$$

is to be mapped to a sector of the form

$$S := \{w \in \mathbb{C} : \phi_1 < \arg(w) < \phi_2\}.$$

Give adequate transformations.

b) Does your method for solving part a)i) also work for the ellipse

$$E := \left\{ z = x + iy \in \mathbb{C} : \frac{x^2}{25} + \frac{y^2}{9} \le 1 \right\}?$$

Hint: Inverse of the Joukowski function.

Solution:

a) (i) The lengths of the half axes $a = \frac{5}{4}$ and $b = \frac{3}{4}$ fulfill $a^2 - b^2 = 1$. The inverse Joukowski functions maps the ellipse to a circle. The radius of that circle is obtained by inserting a point from the boundary of the ellipse. If the correct choice for the is made, it follows that

$$\tilde{f}(z) := z + \sqrt{z^2 - 1}$$

$$\tilde{f}\left(\pm \frac{5}{4}\right) = \pm 2, \qquad \tilde{f}\left(\frac{3i}{4}\right) = 2i.$$

So, a circle with radius 2 around the origin is obtained. To yield the unit circle, we choose

$$f(z) := \frac{1}{2}\left(z + \sqrt{z^2 - 1}\right).$$

Due to $f(0) = -i/2$ the inner part of the ellipse is mapped to the inner part of the unit circle.

(ii) Let $J : \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\} \to \mathbb{C}$ be the Joukowski function on the upper complex half plane. $J^{-1}(z) = z + \sqrt{z^2 - 1}$ maps the hyperbola branches onto rays with

$$\cos(\phi_{1,2}) = \pm\frac{\sqrt{3}}{2},\ \sin(\phi_{1,2}) = \frac{1}{2}\,.$$

As images of the hyperbola branches we obtain the rays $re^{i\frac{\pi}{6}}$ and $re^{i\frac{5\pi}{6}}$. Since $J^{-1}(0) = i$, the area between the hyperbola branches is mapped onto the sector

$$S_1 = \left\{z \in \mathbb{C} : z = re^{i\phi},\ \frac{\pi}{6} < \phi < \frac{5\pi}{6}\right\}.$$

b) No! The following relationships between the half axes a, b and the radius r of the image of the circle hold:

$$a = \frac{1}{2}\left(r + \frac{1}{r}\right), \qquad b = \frac{1}{2}\left(r - \frac{1}{r}\right)$$

so

$$a + b = r \qquad \text{and} \qquad a - b = \frac{1}{r}.$$

With the ellipse for part b) we obtain $a = 5$ and $b = 3$, so

$$a + b = 8 \neq \frac{1}{a - b} = \frac{1}{2}.$$

Not every ellipse around the origin is mapped onto a circle around the origin by the inverse Joukowski function but only those with foci ± 1, or rather $a^2 - b^2 = 1$.

Hand in: 06.05.24 - 10.05.24