Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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Complex functions for Engineering Students
Work sheet 2

Exercise 1:

Find the images of the domains D, $\tilde{D}$ and $\hat{D}$, respectively, regarding the following functions. Sketch the domains and the images or describe them in words.

a) $D = \{z \in \mathbb{C} : |\mathrm{Re}\,(z)| \leq 4, |\mathrm{Im}\,(z)| \leq 2\}$,
$f_1(z) = 0.5z, \quad f_2(z) = 0.5e^{i\frac{\pi}{2}}z$,

b) $\tilde{D} = \{z \in \mathbb{C} : 1 \leq |z| \leq 2, \mathrm{Re}\,(z) > 0, \mathrm{Im}\,(z) < 0\}$,
$f_3(z) = \left(e^{i\frac{\pi}{4}}z\right)^2$, $f_4(z) = \left(e^{i\frac{\pi}{4}}z\right)^2 + 1 + i$, $f_5(z) = \frac{1}{z}$.

c) $\hat{D} := \left\{ z \in \mathbb{C} : z = x + iy, x \in (0,2), y \in \left(0, \frac{\pi}{2}\right)\right\}$,
$f(z) := i \cdot e^z$.

Solution for exercise 1:

a) D: rectangle parralel to the axes with side lengths 8 and 4 and centered around the origin.

D is stretched by f_1 by a factor of 0.5
and in addition f_2 rotates D by $\frac{\pi}{2}$.

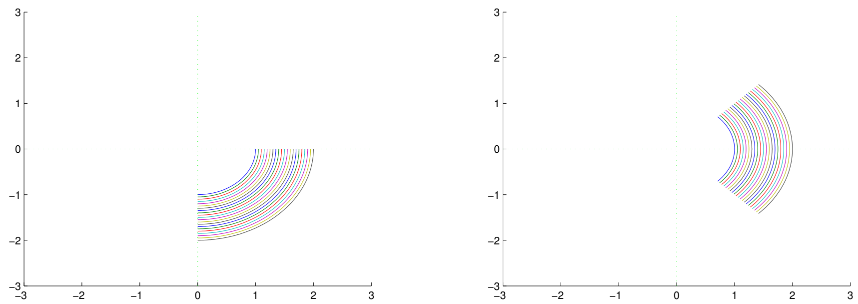
$f_1(D) = \{w \in \mathbb{C} : w = u + iv, -2 \leq u \leq 2, -1 \leq v \leq 1\}$.

$f_2(D) : \{\hat{w} \in \mathbb{C} : \hat{w} = \hat{u} + i\hat{v}, -1 \leq \hat{u} \leq 1, -2 \leq \hat{v} \leq 2\}$.

b) $\tilde{D}$: Lower right fourth of the circular cisk around the origin with inner radius = 1, outer radius = 2. $z = re^{i\phi}$, $r \in [1,2]$, $-\frac{\pi}{2} < \phi < 0$.

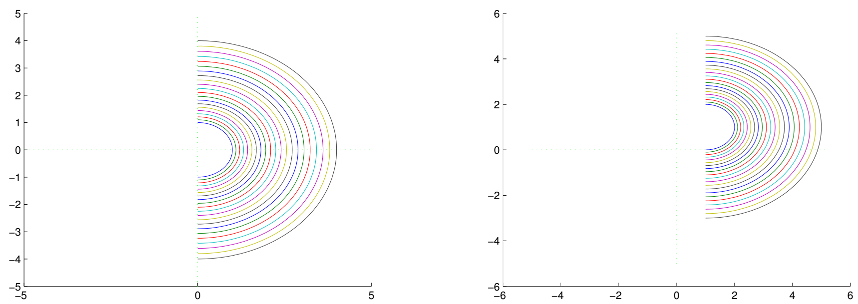
$\tilde{D}$ is rotated by f_3 by $\pi/4$:
$\tilde{w} = e^{i\pi/4}z = \rho\, e^{i\alpha}$, $\rho \in [1,2]$, $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$.

Afterwards, it is squared which yields:
$w = R\, e^{i\beta}$, $R \in [1,4]$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$.



$f_3(\tilde{D})$ is right half of the circular disk around the origin with inner radius $= 1$, outer radius $= 4$.

$f_4(\tilde{D})$ is the transformed image of $\tilde{D}$ by f_3 which then is translated by $1+i$, so die right half of the circular disk around $1+i$ with inner radius $= 1$, outer radius $= 4$.



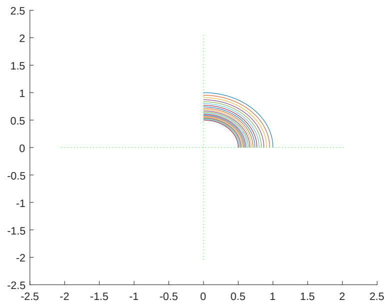
Note: Multiplication of a (constant) complex number results in successive rotation and stretching. Addition of a (constant) complex number results in a translation.

$f_5(\tilde{D})$: $\qquad z = re^{i\varphi} \Longrightarrow \frac{1}{z} = \frac{1}{re^{-i\varphi}}$.

In $\tilde{D}$ it holds that $1 \leq r \leq 2$ and $-\frac{\pi}{2} \leq \varphi \leq 0$. For the images $f_5(z)$ obtained with the points z from $\tilde{D}$ it holds that:

$w = f_5(z) = \rho e^{i\alpha}$ with $\frac{1}{2} \leq \rho \leq 1$ and $0 \leq \alpha \leq \frac{\pi}{2}$.

The image is the upper right quarter of the circular disk around the origin with inner radius 0.5 and outer radius 1.



c) $\hat{D}$: rectangle parralel to the axes with edges $(0,0)$, $(2,0)$, $(2,\frac{\pi}{2})$, $(0,\frac{\pi}{2})$.

$\tilde{f}(z) := e^z \ = \ e^{x+iy} \ = \ e^x \cdot e^{iy} \quad \Longrightarrow$

$\left|\,\tilde{f}(z)\,\right| \ = \ e^x \in (e^0, e^2) \ = \ (1, e^2)\,, \qquad \arg\left(\,\tilde{f}(z)\,\right) \ = \ y \in \left(0, \frac{\pi}{2}\right)$

$f(z) = i \cdot \tilde{f}(z)$: rotated by $\pi/2$. Hence

$|\,f(z)\,| \ = \ \left|\,\tilde{f}(z)\,\right| \ \in \ (1, e^2)$,

$\arg\,(\,f(z)\,) \in \left(\frac{\pi}{2}\,,\,\pi\right)$.

Image: Quarter circular disk in the second quadrant with inner radius 1 und outer radius e^2 .

Exercise 2)

Find all solutions $z \in \mathbb{C}$ of the following equations

i)	$e^z = -1,$	ii)	$e^z = -2\sqrt{2} - 2\sqrt{2}i,$
iii)	$z^5 = 32,$	iv)	$z^5 = 16(1 + i\sqrt{3}),$

Solution: i) $e^z = e^x \cdot e^{iy} = -1$

$|e^z| = e^x = 1 \iff x = 0.$

$e^{iy} = -1 = e^{i\pi} \iff y = \pi + 2k\pi, \quad k \in \mathbb{Z}$

ii) $e^z = -2\sqrt{2} - 2\sqrt{2}i \implies |e^z| = \sqrt{(-2\sqrt{2})^2 + (-2\sqrt{2})^2} = \sqrt{8+8} = 4\,.$

Analogous to i) $x = \ln(4)$.

$e^z = e^x \cdot e^{iy} = -2\sqrt{2} - 2\sqrt{2}i = 4\,(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot i) = 4e^{-i\frac{\pi}{4}}\,.$

$\implies e^{iy} = e^{-i\frac{\pi}{4}} \implies y = 2k\pi - \frac{\pi}{4}, \quad k \in \mathbb{Z}\,.$

This yields infinitely many points of the complex plane that all lie on a line parallel to the imaginary axis.

iii) $z^5 = 32 \iff |z^5| = |z|^5 = 32 \implies |z| = 2$

For $z = re^{i\phi}$ it holds that

$z^5 = r^5 e^{i5\phi} = 2^5 e^0 \implies 5\phi = 0 + 2k\pi \implies \phi = \frac{2k\pi}{5}\,, \qquad k \in \mathbb{Z}.$

This leads to five different points in the complex plane that all lie on the circle with radius two around the origin. The phases are $\{\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, \frac{10\pi}{5}\}$.

iv) $z^5 = 16(1 + i\sqrt{3})$

$|z^5| = |z|^5 = \sqrt{16^2 + 3 \cdot 16^2} = \sqrt{4 \cdot 16^2} = 32 \implies |z| = 2$

For $z = re^{i\phi}$ it holds that

$z^5 = r^5 e^{i5\phi} = 2^5(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 32e^{i\frac{\pi}{3}}$

$\implies 5\phi = \frac{\pi}{3} + 2k\pi \implies \phi = \frac{\pi}{15} + \frac{2k\pi}{5}, \qquad k \in \mathbb{Z}.$

These are again five points around that circle with radius two around the origin. The points are the same as in part iii) except that they are rotated by $\frac{\pi}{15}$.

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