Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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Complex functions for Engineering Students
Homework sheet 2

Exercise 21:

For the $\mathbb{R}^2$, there is a affine linear transformation form any arbitrary rectangle to any arbitrary parallelogram. Check whether the square

$$Q := \{z \in \mathbb{C}, z = x + iy,\, x, y \in [-\sqrt{2}, \sqrt{2}],\, i^2 = -1\}$$

can be transformed (affine linear) to parallelograms with the following corners in $\mathbb{C}$ and if so give an adequate transformation.

a) $\begin{pmatrix} -1 \\ -i \end{pmatrix}, \begin{pmatrix} 3 \\ -i \end{pmatrix}, \begin{pmatrix} 3 \\ i \end{pmatrix}, \begin{pmatrix} -1 \\ i \end{pmatrix}$,

b) $\begin{pmatrix} -1 \\ -i \end{pmatrix}, \begin{pmatrix} 3 \\ -i \end{pmatrix}, \begin{pmatrix} 3 \\ 3i \end{pmatrix}, \begin{pmatrix} -1 \\ 3i \end{pmatrix}$,

c) $\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 2 \\ 2i \end{pmatrix}, \begin{pmatrix} -1 \\ i \end{pmatrix}, \begin{pmatrix} -2 \\ -2i \end{pmatrix}$,

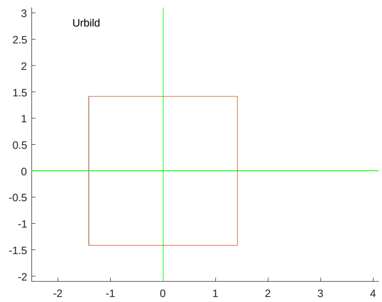
d) $\begin{pmatrix} 0 \\ -2i \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2i \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}$,

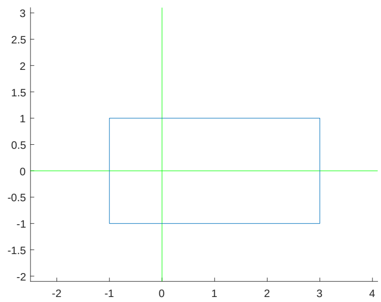
e) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2i \end{pmatrix}, \begin{pmatrix} 2 \\ 4i \end{pmatrix}, \begin{pmatrix} 0 \\ 2i \end{pmatrix}$.

Hint: Sketches can be very beneficial.

Solution:

a) $\begin{pmatrix} -1 \\ -i \end{pmatrix}, \begin{pmatrix} 3 \\ -i \end{pmatrix}, \begin{pmatrix} 3 \\ i \end{pmatrix}, \begin{pmatrix} -1 \\ i \end{pmatrix}$.





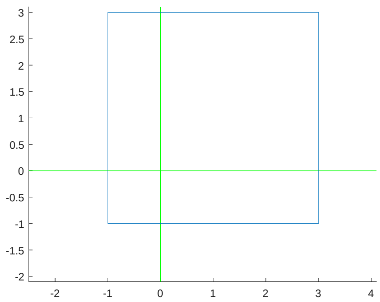
One direction has been stretched differently than the other. Hence, this cannot be represented by an affine transformation in $\mathbb{C}$.

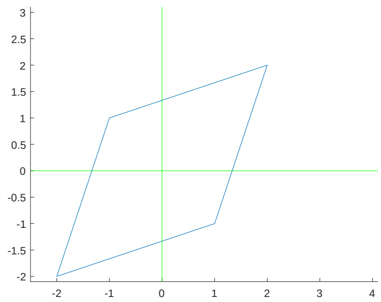
b) $\begin{pmatrix} -1 \\ -i \end{pmatrix}, \begin{pmatrix} 3 \\ -i \end{pmatrix}, \begin{pmatrix} 3 \\ 3i \end{pmatrix}, \begin{pmatrix} -1 \\ 3i \end{pmatrix}$.

$f_b(z) = \sqrt{2}z + 1 + i$ yields the correct transformation.

c) $\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 2 \\ 2i \end{pmatrix}, \begin{pmatrix} -1 \\ i \end{pmatrix}, \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

While the sides are of equal length, the transformation is impossible since successive rotation and stretching cannot change right angles to acute/obtuse angles.





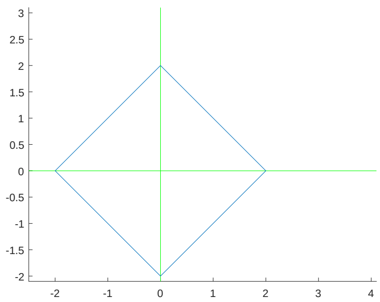
d) $\begin{pmatrix} 0 \\ -2i \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2i \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

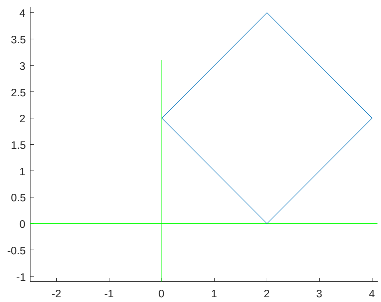
We have a rotation which can be achieved by a linear function, such as:

$f_d(z) := e^{i\frac{\pi}{4}} z$.

e) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 2i \end{pmatrix}$, $\begin{pmatrix} 2 \\ 4i \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2i \end{pmatrix}$.

One can choose $f_e(z) := f_d(z) + 2 + 2i$ (rotation with translation afterwards).





Exercise 2:

Let i be the imaginary unit. Find all complex solutions for the following equations

$$\text{a) } e^{3z} - \frac{i}{e^z} = 0 \qquad \text{bzw.} \qquad \text{b) } e^{2z+1+i\frac{\pi}{2}} = \frac{1}{\sqrt{2}}(1+i)\,.$$

Solution:

a)

$$e^{3z} - \frac{i}{e^z} = 0 \iff e^{4z} = i = 1 \cdot e^{i\frac{\pi}{2}}.$$

Both equations have to be satisfied:

$|e^{4z}| = |e^{4x} \cdot e^{4iy}| = e^{4x} \stackrel{!}{=} |i| = 1$

and

$e^{i4y} \stackrel{!}{=} e^{i\frac{\pi}{2}} \iff 4y = \frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}\,.$

So $x = 0$ and $y_k = \frac{\pi}{8} + \frac{k\pi}{2}, \qquad k \in \mathbb{Z}\,.$

b) $w := e^{2z+1+i\frac{\pi}{2}} = e^{2x+1} \cdot e^{i(2y+\frac{\pi}{2})}. \qquad .$

$|w| = e^{2x+1} \stackrel{!}{=} \left|\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$\iff 2x + 1 = \ln(1) = 0 \iff x = -\frac{1}{2}\,. \qquad .$

$e^{i(2y+\frac{\pi}{2})} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \iff \arg\left(e^{i(2y+\frac{\pi}{2})}\right) = \arg\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + 2k\pi$

$\iff 2y + \frac{\pi}{2} = \frac{\pi}{4} + 2k\pi \iff 2y = -\frac{\pi}{4} + 2k\pi \iff y = -\frac{\pi}{8} + k\pi \qquad k \in \mathbb{Z}.$

Exercise 3: (Please read the hints at the end of the exercise)

Given a transformation $w = f(z) := \dfrac{1}{z}$ mit $z \neq 0$.

a) Find the images of

   (i) the ray $\arg(z) = \varphi_0$,

   (ii) the line $\mathrm{Re}\,(z) = x_0$, so that $z + \overline{z} = 2x_0$,

   (iii) the line $\mathrm{Im}\,(z) = y_0$.

b) Find the image of the circle $|z - \frac{i}{2}| = \frac{1}{2}$ without $z = 0$.

Hints:

1) In all subtasks except a)i), substitute $z = \frac{1}{w}$ into the equations that describe the master images and rearrange these equations so that you can see which quantities are described in the image space. 2) The equation $|z - c| = r$ describes a circle around c with radius r. Be aware that there is the following equivalence that allows for a use without absolute values:

$$|z - c| = R \iff (z-c)\,\overline{(z-c)} = R^2$$

$$\iff (z-c)\,(\overline{z} - \overline{c}) = R^2$$

$$\iff z\overline{z} - z\overline{c} - c\overline{z} + c\overline{c} = R^2\,.$$

Solution:

a) (i) $f : z \to \frac{1}{z} \quad z = re^{i\varphi_0}$

$\frac{1}{z} = \frac{1}{r}e^{-i\varphi_0}$ ray with phase $-\varphi_0$ passing from outside to the inside!

(ii) $\mathrm{Re}\, z = x_0 \iff z + \overline{z} = 2x_0 \iff \frac{1}{w} + \frac{1}{\overline{w}} = 2x_0$

A) $x_0 = 0$
$\frac{1}{z}$ is not defined for $y = x_0 = 0$.
Otherwise, $\frac{1}{z} = \frac{1}{iy} = \frac{-1}{y}i$.
The image is the imaginary axis without zero.

B) $x_0 \neq 0$

$$\frac{1}{w} + \frac{1}{\overline{w}} = 2x_0 \iff \overline{w} + w = 2x_0 w\overline{w} \iff w\overline{w} - \frac{1}{2x_0}\overline{w} - \frac{1}{2x_0}w = 0$$

The image is a circle around $\frac{1}{2x_0}$ with radius $\frac{1}{2|x_0|}$.

(iii) $\mathrm{Im}\,(z) = y_0 \iff z - \overline{z} = 2iy_0$

A) $y_0 = 0$ $\frac{1}{z}$ is not defined for $x = y_0 = 0$.
Otherwise, $z^{-1} = \frac{1}{x}$. The image is the real axis without zero.

B) $\mathrm{Im}\,(z) = y_0 \neq 0$

$$\iff \quad \tfrac{1}{2i}(z - \overline{z}) = y_0$$

$$z - \overline{z} = 2iy_0 \quad \Rightarrow \quad \tfrac{1}{w} - \tfrac{1}{\overline{w}} - 2iy_0 = 0$$

$$w\overline{w} - \tfrac{1}{2iy_0}\overline{w} + \tfrac{1}{2iy_0}w = 0$$

$$\iff \quad w\overline{w} + \tfrac{i}{2y_0}\overline{w} - \tfrac{i}{2y_0}w + \tfrac{1}{4y_0^2} = \tfrac{1}{4y_0^2}$$

The image is a circle around $\frac{-i}{2y_0}$ with radius $\frac{1}{|2y_0|}$.

b) The circle $\left|z - \frac{i}{2}\right| = \frac{1}{2}$ passes through the origin. For $z = i$ $\frac{1}{z} = -i$ is obtained. For $z \to 0$ $\frac{1}{z} \to \infty$ is obtained.

It holds that $z\overline{z} - \frac{i}{2}\overline{z} + \frac{i}{2}z = 0$ or for $z \neq 0$

$$\frac{1}{w\overline{w}} - \frac{i}{2\overline{w}} + \frac{i}{2w} = 0 \iff 1 - \frac{i}{2}w + \frac{i}{2}\overline{w} = 0$$

$$\iff 1 = \frac{i}{2}(w - \overline{w}) = -\mathrm{Im}\,(w)$$

The image is an axis parallel to the real axis passing through $-i$.

Hand in: 22.04.24 - 26.04.24