Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
Prof. Dr. J. Struckmeier,
Dr. H.P. Kiani, T. Hassan

Complex functions for Engineering Students
Work sheet 1

Excercise 1:

a) Rewrite these complex numbers into polar representation ($z = re^{i\phi}$) and draw a sketch of them in the complex plane.

$z_0 = -4, \quad z_1 = \sqrt{8}(-1-i), \quad z_2 = -4i, \quad z_3 = \sqrt{8}(1-i), \quad z_4 = 4,$
$\tilde{z}_k = (-i)^k, \quad k \in \mathbb{N}_0.$

b) Rewrite these complex numbers into cartesian representation ($z = x + iy$).

$z_5 = 3\,e^{i\frac{\pi}{3}}, \quad z_6 = 2e^{i\frac{-\pi}{6}}, \quad z_7 = 2e^{i\frac{-13\pi}{6}}.$

Solution hints for excercise 1:

a) $z_0 = -4 = 4e^{i\pi}$. since
$r = \sqrt{4^2+0^2} = 4, \quad \cos(\phi) = -1, \sin(\phi) = 0 \implies \phi = \pi\,(+2k\pi)\,,$

$z_1 = \sqrt{8}(-1-i) = 4e^{i\frac{-3\pi}{4}}$ since
$r = \sqrt{(\sqrt{8})^2+(\sqrt{8})^2} = 4, \quad \cos(\phi) = \sin(\phi) = -\frac{\sqrt{8}}{r} = -\frac{\sqrt{8}}{4} \implies \phi = -\frac{3\pi}{4}\,(+2k\pi)\,,$

$z_2 = -4i = 4e^{i\frac{-\pi}{2}}$ since
$r = \sqrt{0^2+4^2} = 4, \quad \sin(\phi) = -1, \cos(\phi) = 0 \implies \phi = -\frac{\pi}{2}\,(+2k\pi)\,,$

$z_3 = \sqrt{8}(1-i) = 4e^{i\frac{-\pi}{4}}$ since
$r = \sqrt{(\sqrt{8})^2+(\sqrt{8})^2} = 4, \quad \cos(\phi) = -\sin(\phi) = \frac{\sqrt{2}}{2} \implies \phi = \frac{-\pi}{4}\,(+2k\pi)\,,$

$z_4 = 4 = 4e^{0i}$ since
$r = \sqrt{4^2+0^2} = 4, \quad \cos(\phi) = 1, \sin(\phi) = 0 \implies \phi = 0\,(+2k\pi)\,,$

Sketch : The points are aligned on a circle with radius 4 around the origin, starting at -4 rotated by $\pi/4$ (mathematically positive) until 4 is reached.

$$\tilde{z}_k = (-i)^k = \begin{cases} 1 = e^0 & k = 4l\,, \quad l \in \mathbb{Z} \\ -i = e^{-i\frac{\pi}{2}} & k = 4l+1\,, \quad l \in \mathbb{Z} \\ -1 = e^{-i\pi} & k = 4l+2\,, \quad l \in \mathbb{Z} \\ +i = e^{i\frac{\pi}{2}} & k = 4l+3\,. \quad l \in \mathbb{Z} \end{cases}$$

Sketch : The points are aligned on a circle with radius 1 around the origin, starting at 1 rotated by $\pi/2$ (mathematically negative).

b) $z_5 = 3\,e^{i\frac{\pi}{3}} \; = \; 3\,(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})) \; = \; \frac{3}{2} \, + \, i \cdot \frac{3\sqrt{3}}{2} \, .$

$z_6 = 2e^{i\frac{-\pi}{6}} \; = \; 2\,(\cos(\frac{-\pi}{6}) + i\sin(\frac{-\pi}{6})) \; = = \; 2\,(\cos(\frac{\pi}{6}) - i\sin(\frac{\pi}{6})) \; = \; \sqrt{3} - \, i$

$z_7 == 2e^{i\frac{-13\pi}{6}} \; = \; 2\,(\cos(\frac{-13\pi}{6}) + i\sin(\frac{-13\pi}{6})) \; = \; 2\,(\cos(\frac{-\pi}{6}) + i\sin(\frac{-\pi}{6})) \; = \; z_6 \, .$

Excercise 2:

Let $z_1, \ldots, z_6$ be defined as in exercise 1. Calculate the Cartesian representations of the following complex numbers.

$$\mathrm{Re}\,(z_1), \quad \mathrm{Im}\,(z_1), \quad \mathrm{Re}\,(z_3), \quad \mathrm{Im}\,(z_3)\,, \quad z_1 + z_3\,, \quad z_1 - z_3\,,$$

$$2z_5 + \sqrt{8}z_3\,, \quad \bar{z_1}\,, \quad z_1 \cdot \bar{z_1}\,, \quad z_1 \cdot z_2\,, \quad (z_6)^2 \cdot (z_5)^4\,, \quad \frac{z_5}{z_6}\,.$$

Solution hints for excercise 2:

$$\mathrm{Re}\,(z_1) = \mathrm{Re}\,\left(\sqrt{8}(-1-i)\right) = -\sqrt{8}\,. \qquad \mathrm{Im}\,(z_1) = -\sqrt{8}\,.$$

$$\mathrm{Re}\,(z_3) = \mathrm{Re}\,\left(\sqrt{8}(1-i)\right) = \sqrt{8}\,. \qquad \mathrm{Im}\,(z_3) = -\sqrt{8}\,.$$

$$z_1 + z_3 = -2\sqrt{8}\,i\,. \qquad z_1 - z_3 = -2\sqrt{8}.$$

$$2z_5 + \sqrt{8}z_3 = 2\left(\tfrac{3}{2} + i \cdot \tfrac{3\sqrt{3}}{2}\right) + \sqrt{8}\left(\sqrt{8}(1-i)\right) = (3+8) + i(3\sqrt{3}-8) = 11 + i(3\sqrt{3}-8)\,.$$

$$\bar{z_1} = -\sqrt{8} + i\sqrt{8} \qquad z_1 \cdot \bar{z_1} = 4e^{i\frac{-3\pi}{4}} \cdot 4e^{i\frac{3\pi}{4}} = 4 \cdot 4 \cdot e^0 = |z_1|^2 = 4^2 = 16\,.$$

$$z_1 \cdot z_2 = 4e^{i\frac{-3\pi}{4}} \cdot 4e^{i\frac{-\pi}{2}} = 16e^{i\frac{-5\pi}{4}} = 16e^{i\frac{3\pi}{4}} = 16(\cos(\tfrac{3\pi}{4}) + i\sin(\tfrac{3\pi}{4})) = 8\sqrt{2}(-1+i)$$

As an exception, the multiplication in cartesian representation is not costlier (calculation-wise) compared to polar representation:

$$z_1 \cdot z_2 = \sqrt{8}(-1-i) \cdot (-4i) = 4\sqrt{8}(i + i^2) = 8\sqrt{2}(-1+i)$$

$$(z_6)^2 \cdot (z_5)^4 = \left(2e^{i\frac{-\pi}{6}}\right)^2 \left(3\,e^{i\frac{\pi}{3}}\right)^4 = 2^2 \cdot 3^4 \cdot e^{\frac{-2i\pi}{6} + \frac{4i\pi}{3}} = 4 \cdot 81 \cdot e^{i(\frac{4\pi}{3} - \frac{\pi}{3})} = 324e^{i\pi} = -324\,.$$

$$\frac{z_5}{z_6} = \frac{3\,e^{i\frac{\pi}{3}}}{2e^{i\frac{-\pi}{6}}} = \tfrac{3}{2}e^{i(\frac{\pi}{3} + \frac{\pi}{6})} = \tfrac{3}{2}e^{i\frac{\pi}{2}} = \tfrac{3}{2}\,i\,.$$

Excercise 3:

Characterize these subsets of the complex plane by sketch or explanation:

$$\begin{aligned} M_1 &= \{z \in \mathbb{C} \mid |z+4-3i| \leq 5\}, \\ M_2 &= \{z \in \mathbb{C} \mid |z-i| = |z-2-i|\}\,, \\ M_3 &= \{z \in \mathbb{C} \mid z+\bar{z} = 2\}\,, \\ M_4 &= \{0\} \cup \left\{z \in \mathbb{C}\setminus\{0\} \mid \mathrm{Re}\left(\frac{z}{\bar{z}}\right) = 0\right\}. \end{aligned}$$

Solution for excercise 3:

a) M_1: circular disk with radius $r = 5$ und center $M = -4+3i$ including the boundary.

b) M_2: distance z to i = distance z to $2+i$ $\Rightarrow$ perpendicular bisector between i and $2+i$.
which is the line $\{x+iy\ \mathbb{C}\ :\ x = 1\}$.

c) M_3: $\mathrm{Re}\,(z) = 1$: line parallel to the imaginary axis going through $1 + 0\cdot i$.

d) M_4: $\mathrm{Re}\left(\frac{z}{\bar{z}}\right) = \mathrm{Re}\left(\frac{z^2}{z\bar{z}}\right) = 0$

With $z \neq 0$ it follows that: $\mathrm{Re}\left(\frac{(x+iy)^2}{(x+iy)(x-iy)}\right) = 0 \iff x^2 - y^2 = 0$

M_4 consists of the two lines $\mathrm{Re}\,(z) = \pm\mathrm{Im}\,(z)$.

alternatively: $z = re^{i\phi},\ r \neq 0$.

$\mathrm{Re}\left(\frac{z}{\bar{z}}\right) = \mathrm{Re}\left(\frac{re^{i\phi}}{re^{-i\phi}}\right) = \mathrm{Re}\left(e^{i2\phi}\right) = 0 \iff e^{i2\phi} = \pm i$

Hence $2\phi = \pm\frac{\pi}{2} \qquad (+2k\pi \qquad k \in \mathbb{Z})$

$\iff \phi = \pm\frac{\pi}{4} \qquad (2k\pi \qquad k \in \mathbb{Z})$

Together with the origin, we again obtain the two straight lines $\mathrm{Re}\,(z) = \pm\mathrm{Im}\,(z)$.

Excercise 4: Describe the following subsets of the complex number plane using formulas similar to those in task 3.

M_5 : strip parallel to the imaginary axis with a width of 4, symmetric to $z_0 = 1 + i$, including the boundary.

M_6 : circular disk around the origin with inner radius 1 and outer radius 3, without boundary.

M_7 : circular disk (punctured disk) around the origin with inner radius 0 and outer radius 3, without boundary.

M_8 : sector between the lines $\mathrm{Re}\,(z) = \mathrm{Im}\,(z)$ and $-\mathrm{Re}\,(z) = \mathrm{Im}\,(z)$ in the upper half-space, without boundary.

Solution:

$$M_5 = \{z \in \mathbb{C} \mid -1 \leq \mathrm{Re}\,(z) \leq 3\},$$

$$M_6 = \{z \in \mathbb{C} \mid 1 < |z| < 3\},$$

$$M_7 = \{z \in \mathbb{C} \mid 0 < |z| < 3\},$$

$$M_8 = \{z \in \mathbb{C} \mid z = re^{i\phi},\, r > 0,\, \frac{\pi}{4} < \phi < \frac{3\pi}{4}\}.$$

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