Fachbereich Mathematik der Universität Hamburg **SoSe 2024**
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Complex functions for Engineering Students

Work sheet 4

Exercise 1:

a) Determine a Möbius transform $T : \mathbb{C}^* \to \mathbb{C}^*, \quad T(z) := \dfrac{az+b}{cz+d}$ with

$$T(i) = 0\,, \qquad T(0) = 2\,, \qquad T(2i) = \infty\,.$$

b) Determine the images of the following generalized circles using T.

(i) $K :=$ imaginary axis,

(ii) $K_2 := \{\, z \in \mathbb{C} \,:\, |z| = 2 \,\}$,

(iii) $\tilde{K} :=$ real axis.

c) Determine the image of the quarter plane

$$S := \{\, z \in \mathbb{C} \,:\, \mathrm{Re}\,(z) > 0,\, \mathrm{Im}\,(z) > 0 \,\}\,.$$

d) Determine the image of

$$H := \{\, z \in \mathbb{C} \,:\, \mathrm{Re}\,(z) > 3 \,\}\,.$$

Exercise 2:

In which points of their domain are the following functions complex differentiable?

a) $f_1 : \mathbb{C} \to \mathbb{C},\ f_1(z) = \mathrm{Re}\,(z) \cdot \mathrm{Im}\,(z)\,.$

b) $f_2 : \mathbb{C} \to \mathbb{C},$
$f_2(z) = (\mathrm{Re}\,(z)+2)^2 - (\mathrm{Im}\,(z)+2)^2 + i\Big[\mathrm{Im}\,(z)(\mathrm{Re}\,(z)+4) + \mathrm{Re}\,(z)(\mathrm{Im}\,(z)+4)\Big]\,.$

c) $f_3 : \mathbb{C}\setminus\{0\} \to \mathbb{C},\ f_3(z) = \dfrac{z^2}{\bar{z}}\,.$
Hint: Use Cauchy Riemann equations in polar representation: $u_r = \frac{1}{r}v_\varphi$ and $v_r = -\frac{1}{r}u_\varphi\,.$

Exercise 3:

Hint : You do not need to give exact transformations.

a) To solve a potential problem, the area outside the two circles
$K_1 := \left\{ z \in \mathbb{C} \,:\, |z-2| \,\leq\, \frac{3}{2} \right\}$, and
$K_2 := \left\{ z \in \mathbb{C} \,:\, |z+1| \,\leq\, \frac{3}{2} \right\}$
is to be mapped onto a strip parallel to or onto the interior of the circle around the origin. Which of these two transformations is achievable by the use of a Möbius transform?

b) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ be fixed parameters. Which of the following areas can be mapped onto a sector of the form

$$S := \left\{ w \in \mathbb{C} \,:\, w = r\,e^{i\phi},\, r \in \mathbb{R}^+,\, -\pi \,<\, \varphi_1 \,<\, \phi \,<\, \varphi_2 \,<\, \pi \right\}$$

using Möbius transforms? Please explain your answer.

(i)
$$G_1 := \{\, z \in \mathbb{C} \,:\, \alpha < |z| < \beta \} \,.$$

(ii)
$$G_2 := \{\, z \in \mathbb{C} \,:\, \alpha < \mathrm{Re}\,(z) \,<\, \beta \} \,.$$

(iii)
$$G_3 := \left\{ z \in \mathbb{C} \,:\, |z - \alpha| < \frac{3}{4}\,|\beta - \alpha|\,,\, |z - \beta| < \frac{3}{4}\,|\beta - \alpha| \right\} \,.$$

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