Complex Functions: Auditorium Exercise-02

For Engineering Students

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Addition in $\mathbb{R}^2 \longleftrightarrow \mathbb{C}$

Reminder: Addition

$z = x + iy \in D \subset \mathbb{C}, \qquad c = a + ib \in$ fixed.

$f : D \to \mathbb{C}, \qquad f : z \mapsto z + c = (x + a) + i(y + b)$

Geometric Interpretation:
The subset D of the complex plane is shifted by c.

Multiplication in $\mathbb{R}^2 \longleftrightarrow \mathbb{C}$

Multiplication

$z = re^{i\varphi} \in D \subset \mathbb{C}, \qquad c = \rho e^{i\alpha} \in$ fixed.

$f(z) = c \cdot z = re^{i\varphi} \cdot \rho e^{i\alpha} = \rho r e^{i\varphi} e^{i\alpha}$

- The magnitudes are multiplied: ρr.
- The arguments are added: $\alpha + \varphi$

Example A: Rectangle under affine linear function

$D := \{z \in \mathbb{C} : -3 < \mathrm{Re}(z) < 0, \, 0 < \mathrm{Im}(z) < 6\}$

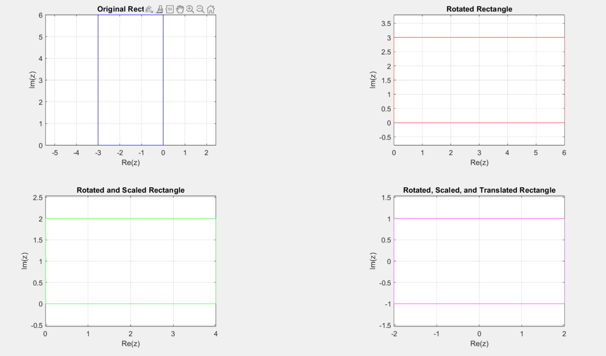
$f(z) := \frac{2}{3} e^{-i\frac{\pi}{2}} z - (2 + i)$

Geometric Solution:

- The rectangle is rotated by $\frac{\pi}{2}$ in a mathematically negative direction (clockwise),
- scaled by a factor of $\frac{2}{3}$,
- and shifted by $c = -2 - i$.

Example A:

Geometric Solution:



Computational:
Provide ranges for u, v or r, φ for $f(z) = u + iv$ or $f(z) = re^{i\varphi}$

$$f(z) = u + iv = \quad = \frac{2}{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)(x + iy) - (2 + i)$$

Solution:

$$\begin{aligned} f(z) = u + iv &= \frac{2}{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)(x+iy) - (2+i) \\ &= \frac{2}{3}(-i)(x+iy) - (2+i) \\ &= (\frac{2}{3}y - 2) + i(-\frac{2}{3}x - 1) \end{aligned}$$

$$\boxed{-2 < u < 2 \quad -1 < v < 1}$$

Geometric Interpretation of affine linear function

$$f(z) = cz + d$$

Rotation and scaling + subsequent translation

- Scaling is uniform in all directions
- Angles are preserved

Difference from linear mappings in $\mathbb{R}^2$:

$$\text{for example:} \qquad \mathbf{v} \mapsto \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix} \mathbf{v}$$

Scaling by 1/2 in the $x-$direction and by 3 in the $y-$direction

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Longrightarrow \mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Angle between pre-images $= \frac{\pi}{2}$,

Angle between images $= \frac{\pi}{4}$.

In **complex multiplication** $c \cdot z$,

- only rotation and stretching occur
- Angles remain preserved.

Linear functions in $\mathbb{C}$ can do much less than linear functions in $\mathbb{R}^2$

Differentiability in $\mathbb{C}$: implies more than differentiability in $\mathbb{R}^2$.

Geometric Interpretation of affine linear function

Locally: $f(z) =: u(x,y) + i\,v(x,y) \approx az + b$

In $\mathbb{R}^2$: $\tilde{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \implies J\tilde{f}(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

Rotation and scaling in $\mathbb{R}^2$: $\begin{pmatrix} x \\ y \end{pmatrix} \to \mathbf{A} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ with

$$\mathbf{A} = \begin{pmatrix} k\cos(\alpha) & -k\sin(\alpha) \\ k\sin(\alpha) & k\cos(\alpha) \end{pmatrix}$$

Next: Differentiability in $\mathbb{C}$:

$$u_x = v_y, \quad v_x = -u_y$$

(Cauchy-Riemann equations).

The function $f : D \to \mathbb{C}$, $w = f(z) := z^{-1}$

Pointwise calculation $\longrightarrow$ easier in polar coordinates.

Example B: Image of the sector

$$D := \{\, z \in \mathbb{C} \,:\, z = re^{i\phi} : \phi \in]\tfrac{\pi}{6}, \tfrac{\pi}{3}[, r \in \mathbb{R}^+\}$$

Let $\rho e^{i\alpha}$ be the polar coordinate representation of $w := f(z)$.
Then we have

$$w := f(z) = \rho e^{i\alpha} = \frac{1}{re^{i\phi}} = \frac{1}{r} e^{-i\phi}$$

$$\Longrightarrow \rho = \tfrac{1}{r} \in]0, \infty[, \qquad \alpha = -\phi \in] - \tfrac{\pi}{3}, -\tfrac{\pi}{6}[$$

The image is the sector mirrored at the real axis.

Example C:
D = Image of the circle with radius 2 around $z_0 = 2$ excluding zero.

$$D = \{\, z \in \mathbb{C} \setminus \{0\} \;:\; |z - 2| = 2\}$$

Example C

Describe the circle without absolute value signs:

$$|z-2|^2 = (z-2)(\overline{z-2}) \;=\; z\bar{z} - 2\bar{z} \;-2z + 4 = 4$$

$$\iff z\bar{z} - 2\bar{z} \;-2z = 0,$$

set $z = \frac{1}{w}$ and rearrange until it is clear what we have:

$$\frac{1}{w}\frac{1}{\bar{w}} - 2\frac{1}{\bar{w}} \;-2\frac{1}{w} = 0 \qquad \text{multiply by } w\bar{w} \neq 0$$

$$\iff 1 - 2w - 2\bar{w} = 1 - 2(w+\bar{w}) = 1 - 4\,\mathrm{Re}(w) = 0$$

So $\mathrm{Re}(w) = \frac{1}{4}$. The image is the line parallel to the imaginary axis passing through the point $z = \frac{1}{4}$.

Natural Powers and Roots

$$z = re^{i\varphi} \implies z^n = r^n(e^{i\varphi})^n = r^n(e^{in\varphi})$$

- Magnitude: raised to the power of n,
- Angle: multiplied by n.

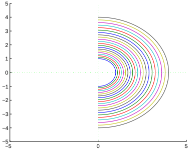
Example D: $D = \{z \in \mathbb{C} : z = re^{i\varphi}, r \in [1,2], -\frac{\pi}{4} < \phi < \frac{\pi}{4}\}$

$w = \rho e^{i\alpha} = f(z) = z^2 = \left(re^{i\varphi}\right)^2 = r^2 e^{i2\varphi}$

$\implies |w| =$

Angle of $(w) = \arg(w) =$

So $f(D) = \{w \in \mathbb{C} : w = \rho e^{i\alpha}, \rho \in \qquad\qquad, \alpha \in \qquad\qquad \}$



Inverse: $w = \sqrt[n]{z} = \sqrt[n]{r}\, e^{i\,\frac{\varphi}{n}}$

CAUTION!!

$$w^n = z \iff (w = \rho e^{i\alpha})^n = re^{i\varphi} \iff \rho^n e^{in\alpha} = re^{i\varphi}$$

It does **not** follow: $n\alpha = \varphi$, but

$$\rho^n = r \qquad \text{and} \qquad e^{in\alpha} = e^{i\varphi}$$

Example E: The second roots of i are sought.

$$w = \rho \cdot e^{i\alpha} = \sqrt[2]{i} \implies w^2 = i = 1 \cdot e^{i\frac{\pi}{2}}$$

$$w^2 = \rho^2 \cdot e^{i2\alpha} \stackrel{!}{=} 1^2 \cdot e^{i\frac{\pi}{2}}$$

Visually:
w lies on the unit circle and if we double the angle to w you land on i.

Computationally: $\rho^2 \stackrel{!}{=} 1^2 \underset{\rho>0}{\implies} \rho = 1$

$$e^{i2\alpha} \stackrel{!}{=} e^{i\frac{\pi}{2}} \implies$$

These are infinitely many possible representations of two different points on the complex plane.

General: $n \in \mathbb{N}, \quad z = re^{i\phi}, \quad w := z^{\frac{1}{n}}$

$$w^n = (\rho \cdot e^{i\alpha})^n = \rho^n \cdot e^{in\alpha} \overset{!}{=} |z|e^{i\phi} \iff \rho = |z|^{\frac{1}{n}} \wedge e^{in\alpha} = e^{i\phi}$$

$$\implies n\alpha = \phi + 2k\pi \iff \alpha = \frac{\phi}{n} + \frac{2k\pi}{n},\ k \in \mathbb{Z}.$$

With $\quad w = |z|^{\frac{1}{n}} e^{i(\frac{\phi}{n} + \frac{2k\pi}{n})} \qquad k = 0, 1, 2, \cdots, n-1$

we obtain the n pairwise different points in the $\mathbb{C}-$ plane, for which $w^n = z$ holds.

As **principal value** we define: $z^{\frac{1}{n}} = |z|^{\frac{1}{n}} e^{i(\frac{\phi}{n})}$, $-\pi < \phi < \pi$.

Example F: Find all solutions $z = re^{i\phi}$ of

$$\underbrace{\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}_{\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)} z^4 = \underbrace{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}_{\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)} \cdot 81$$

Solution:

$$e^{i\frac{\pi}{3}} z^4 = 81 e^{\frac{i2\pi}{3}} \iff z^4 = r^4 e^{i4\phi} = 81 e^{i\frac{\pi}{3}}$$

$$r^4 = |z|^4 = 81 \iff r = |z| = 3$$

$$e^{i4\phi} = e^{i\frac{\pi}{3}} \iff 4\phi = 2k\pi + \frac{\pi}{3}$$

$$\{\arg(z)\} = \{\frac{\pi}{12}, \frac{\pi}{12} + \frac{2\pi}{4}, \frac{\pi}{12} + \frac{4\pi}{4}, \frac{\pi}{12} + \frac{6\pi}{4}, \cdots\}$$

These are infinitely many representations, but only four different points in the complex plane.

Illustration of the Exponential Function:

$$\boxed{\exp(z) = e^x \cdot e^{iy} = e^x(\cos(y) + i\sin(y))}$$

Determine the image of the coordinate grid

It holds (as above)

$$|\, e^z \,| = e^{\mathrm{Re}(z)}, \qquad \arg(e^z) = \mathrm{Im}(z) \qquad (+2k\pi)$$

Case 1: $z = x_0 + iy$ with x_0 fixed and $y \in \mathbb{R}$

$$e^{x_0+iy} = e^{x_0} \cdot e^{iy} \ : \text{Magnitude fixed, Argument varies}$$

Image: Circle traversed infinitely many times with radius $R_{x_0} = e^{x_0}$
Stripes parallel to the $y-$axis $\rightarrow$ Ring

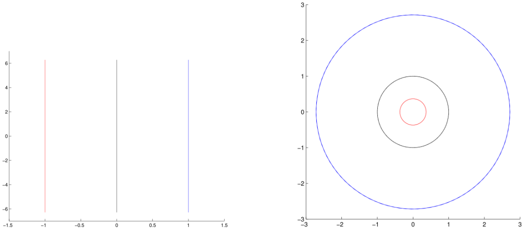


Figure: Exponential function

Case 2: $z = x + iy_0$ with y_0 fixed and $x \in \mathbb{R}$

$$e^{x+iy_0} = e^x \cdot e^{iy_0} \ : \text{Magnitude varies, Argument fixed}$$

Image: Ray with angle $\alpha = y_0$

Stripes parallel to the $x-$axis $\to$ Sector

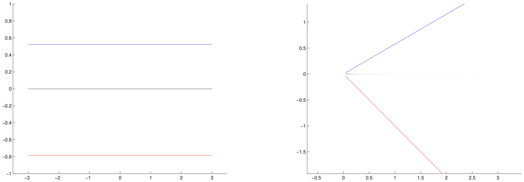


Figure: Exponential function

Example G:

Find the solutions of

- $e^z = -2$
- $(e^z)^2 = -25i$

Solution:

(a) $e^z = e^{x+iy} = e^x \cdot e^{iy} = -2$

$$e^x = 2 \iff x = ln(2)$$

$$e^{iy} = e^{i\pi} \iff y = \pi + 2k\pi$$

$$z = x + iy = ln(2) + i(\pi + 2k\pi)$$

(b) $(\mathrm{e}^z)^2 = e^{2z} = e^{2x+2iy} = e^{2x} \cdot e^{i2y} = -25i$

$|(e^z)^2| = e^{2x} \overset{!}{=} 25 \iff e^x = 5 \iff x = \ln(5).$

$e^{i2y} = -i = e^{-i\frac{\pi}{2}} \iff 2y = 2k\pi - \frac{\pi}{2} \iff y = k\pi - \frac{\pi}{4}$

These are infinitely many solutions!

Example H: Image of

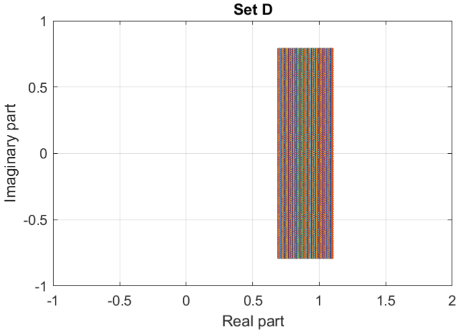
$$D := \left\{ z \in \mathbb{C} \, : \, z = x + iy, \, |x| \in [\ln(2), \ln(3)], \, |y| \leq \tfrac{\pi}{4} \right\}$$

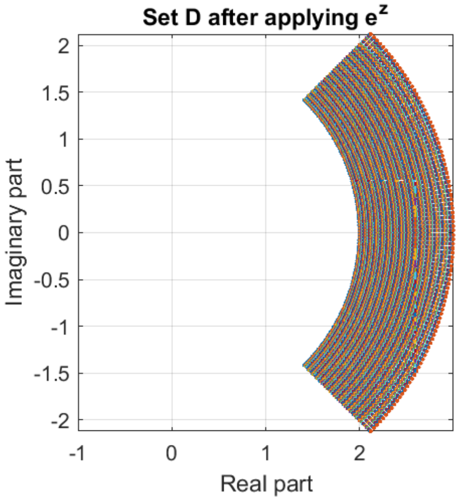
under the mapping

$$f(z) := 2 \cdot e^{i\frac{3\pi}{4}} e^{z}.$$

Sketch:

Example




Set D after applying e^z
Imaginary part
Real part
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2

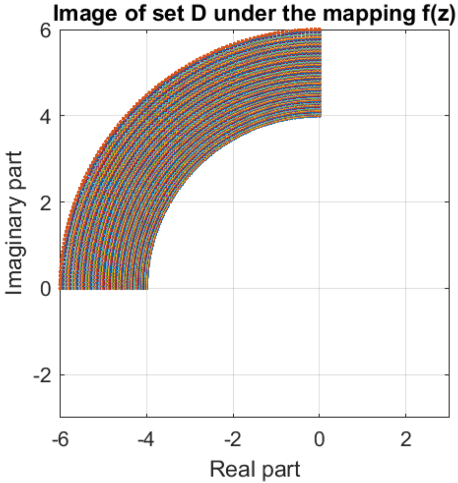


Image of set D under the mapping f(z)
6
4
2
0
-2
Imaginary part
-6
-4
-2
0
2
Real part

THANK YOU