Complex Functions: Auditorium Exercise-01

For Engineering Students

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A Complex Number is a combination of a 'Real Number' and an 'Imaginary Number'.

$$z = x + iy$$

$$\underbrace{z}_{\text{complex number}} = \underbrace{x}_{\text{Real Part}} + i\underbrace{y}_{\text{Imaginary Part}}$$

$x =: \mathrm{Re}(z)$ and $y =: \mathrm{Im}(z)$

Squaring a real number will give us either a +ve number or 0.

$$\boxed{?^2 = -1}$$

A Complex number on a Plane

$z \in \mathbb{C} : z := x + iy, \quad x, y \in \mathbb{R}, \quad i =$ imaginary unit
$\mathbb{C}$ is the field of complex numbers.

Example of a Complex number: $z = 1 + 2i$

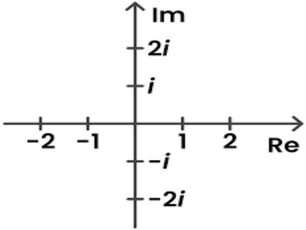

Figure: Complex Plane (Gauß Plane)

We can identify $z = (1, 2) \in \mathbb{C}$ as a point in the Complex Plane.

Addition and Multiplication in $\mathbb{R}^2 \longleftrightarrow \mathbb{C}$

Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Example: $(1 + 2i) + 2i$

Geometric Interpretation:

$f : D \to \mathbb{C}, \qquad f : z \mapsto z + c, \qquad c \in \mathbb{C}$ fixed.

For example, $D := \{z \in \mathbb{C}, 0 \leq \mathrm{Re}(z) \leq 1,\ 0 \leq \mathrm{Im}(z) \leq 1\}$ and $c = 2 + i$

Multiplication:

In $\mathbb{R}^2$ there is no multiplication $v, w \in \mathbb{R}^2, v * w \in \mathbb{R}^2$, only

$$w.v := < w, v > \in \mathbb{R}, \qquad Av \in \mathbb{R}^2, \quad A \in \mathbb{R}^{2\times 2}$$

In the complex plane **define**:

$$(a+ib)(x+iy) := (ax-by) + i(ay+bx) \qquad \in \mathbb{C}$$

Alternatively, using the usual properties of multiplication/addition:

$$(a+ib)(x+iy) = ax + iay + ibx + i^2by \quad and \quad i^2 = -1$$

Complex conjugate: $\bar{z} := \overline{(x+iy)} = (x - iy)$

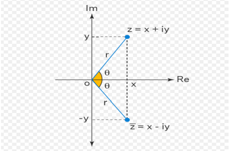

Figure: complex conjugate

For every complex number $z = x + iy$ we can generate its mirror image along the real axis. And we get the Conjugate of the complex number.

Division in the complex plane : $\dfrac{z_1}{z_2} := \dfrac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2}$

Magnitude or Modulus: Distance from the origin $(0 + i \cdot 0)$ on the complex plane

$$= \text{Length of the position vector} \quad = \sqrt{x^2 + y^2}.$$

$$|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z \cdot \bar{z}\,.$$

$|z_1 - z_2| =$ Distance between z_1 and z_2

Examples: What geometric objects are described by

$$\begin{aligned} |z + 2 + i| &= 3, \\ (z - 5)(\bar{z} - 5) &= 4, \\ |z + 2 - 2i| &= |z - 4 - 3i| \end{aligned}$$

Polar Coordinates: as in $\mathbb{R}^2$: $\quad r = \sqrt{x^2 + y^2}$

$x = r\cos(\varphi), \qquad y = r\sin(\varphi)\,,$

$z = r\,(\,\cos(\varphi) + i\,\sin(\varphi)\,)$

$r = |z| =$ Distance from z to Zero

$\varphi =$ Angle between position vector and positive x-axis in the mathematically positive direction
$= \arg(z) =$ **Argument of z.**

$z = x + iy$ given $\Longrightarrow$ Argument determined only up to multiples of 2π!

Given $z = x + iy \neq 0$, the following argument can be chosen:

$$\varphi = \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0 \\ -\pi + \arctan\left(\frac{y}{x}\right) & x < 0 \\ \frac{\pi}{2} & x = 0 \wedge y > 0 \\ -\frac{\pi}{2} & x = 0 \wedge y < 0 \end{cases}$$

Examples: Arguments and magnitudes of

$$z_1 = 4 + i4\sqrt{3}, \quad z_2 = -i, \quad z_3 = -4\sqrt{3} + 4i.$$

Exponential Form or Euler Form :

For $\phi \in \mathbb{R}$, $\boxed{\cos(\phi) + i \cdot \sin(\phi) \ = \ e^{i\phi}}$

Due to $i^4 = (i^2)^2 = (-1)^2 = 1$, we have

$$i^{4k} = 1, \qquad i^{4k+1} = i, \qquad i^{4k+2} = -1 \qquad i^{4k+3} = -i\,.$$

One can define exp, cos, sin using series:

$\exp(y) = \sum_{l=0}^{\infty} \frac{(y)^l}{l!}, \quad \cos(y) = \sum_{m=0}^{\infty} (-1)^k \frac{y^{2m}}{(2m)!}$

$\sin(y) = \sum_{m=0}^{\infty} (-1)^k \frac{y^{2m+1}}{(2m+1)!}$

Thus, (under uniform convergence of the involved series) for all $y \in \mathbb{R}$, we have

$$\begin{aligned}
\exp(iy) &= \sum_{l=0}^{\infty} \frac{(iy)^l}{l!} = \sum_{l=0}^{\infty} \frac{i^l y^l}{l!} \\
&= \sum_{k=0}^{\infty} \left(\frac{i^{4k} y^{4k}}{(4k)!} + \frac{i^{4k+1} y^{4k+1}}{(4k+1)!} + \frac{i^{4k+2} y^{4k+2}}{(4k+2)!} + \frac{i^{4k+3} y^{4k+3}}{(4k+3)!} \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{y^{4k}}{(4k)!} - \frac{y^{4k+2}}{(4k+2)!} \right) + i \cdot \sum_{k=0}^{\infty} \left(\frac{y^{4k+1}}{(4k+1)!} - \frac{y^{4k+3}}{(4k+3)!} \right) \\
&= \sum_{m=0}^{\infty} \left((-1)^k \frac{y^{2m}}{(2m)!} \right) + i \cdot \sum_{m=0}^{\infty} \left((-1)^k \frac{y^{2m+1}}{(2m+1)!} \right)
\end{aligned}$$

Polar Coordinates Using the Exponential Function

So we obtain for $z = x + iy$

$x = r\cos(\varphi), \qquad y = r\sin(\varphi)\,, \quad r = \sqrt{x^2 + y^2}$

$z = r\,(\cos(\varphi) + i\,\sin(\varphi)\,) = re^{i\varphi} = |z|e^{i\varphi}$

Argument of $z = \arg(z) = \arg(re^{i\varphi})$

Attention: Argument determined only up to multiples of 2π, because $e^{i\varphi}$ is 2π periodic!

$$re^{i\varphi} = re^{i(\varphi + 2k\pi)} \quad \forall\, k \in \mathbb{Z}$$

$\{\arg z\}$ or $[\arg z] :=$ Set of all arguments of z

$\arg(z) :=$ Principal value of argument z, determined by additional condition

$$\text{usually} \quad \varphi \in]-\pi, \pi] \quad \text{(Principal value)}$$

$\arg(0)$ is not defined!

A: Polar Coordinates of:

- $z_1 = 4 + i4\sqrt{3}$
- $z_2 = -i$
- $z_3 = -4\sqrt{3} + 4i$

B: Unit Circle:

$$K_1 := \{z \in \mathbb{C} : \; z = re^{i\varphi}, \; r = 1, \; \varphi \in [0, 2\pi)\}$$

$$= \{z \in \mathbb{C} : \; z = e^{i\varphi}, \; \varphi \in [0, 2\pi)\}$$

or $$= \{z \in \mathbb{C} : \; z = e^{i\varphi}, \; \varphi \in (-\pi, \pi]\}$$

C: Imaginary Unit: $|i| = |0 \cdot 1 + 1 \cdot i| =$

How does i look like in polar coordinates?

$$i =$$

$$i^2 =$$

D: Generally, multiplication is simpler in polar form

For example, $z_1 \cdot z_3$ for $z_1 = 4 + i4\sqrt{3}, \quad z_3 = -4\sqrt{3} + 4i.$

$z_1 \cdot z_3 = (4 + i4\sqrt{3})(-4\sqrt{3} + 4i) =$

Calculated above: $z_1 = 8e^{i\frac{\pi}{3}} \quad z_3 = 8e^{i\frac{2\pi}{3}} =$

$z_1 \cdot z_3 = 8e^{i\frac{\pi}{3}} \cdot 8e^{i\frac{2\pi}{3}}$

E: Magnitude e^z :

$|e^z| = |e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x| \cdot |e^{iy}|$

Because $|e^{iy}| \; = \; |\cos(y) + i\sin(y)| =$

$$|\, e^z \,| \; = \; e^x \; = \; e^{\mathrm{Re}(z)}$$

And, $\arg(e^z)$?

Conjugate Complex Number in Polar Form:

$z = re^{i\varphi} \implies \bar{z} =$

Geometrically: Reflection on the real (x)-axis

For $z = x + iy = re^{i\varphi}$ and fixed $c = a + ib = \rho e^{i\alpha}$

Addition: $f : z \mapsto c + z$ cartesian: $z + c = (x + iy) + (a + ib)$
polar: $z + c = re^{i\varphi} + \rho e^{i\alpha} =$
geometric: Translation by c

Multiplication: $f : z \mapsto c \cdot z$ cartesian: $c \cdot z = (x + iy) \cdot (a + ib)$
polar: $c \cdot z = re^{i\varphi} \cdot \rho e^{i\alpha}$
geometric: Next lesson/homework

To conclude, here are a few examples:

A) Sketch/Describe in words the following subsets of the complex plane

I) $D = \{z \in \mathbb{C} : |z - 5i| = 2\}$

II) $D = \{z \in \mathbb{C} : |z - 5i| \leq 2\}$

III) $\tilde{D} := \{z \in \mathbb{C} : 1 < |z - 5i| < 2\}$

IV) $\tilde{D} := \{z \in \mathbb{C} : 0 < |z - 5i| < 2\}$

Example: B) Describe the following subsets of the complex plane using formulas.

M_1: Strip parallel to the imaginary axis with width 6, symmetric to $z_0 = -3 - 2i$, including boundaries.

M_2: Open annulus around $z_0 = -3 - 2i$ with inner radius 2 and outer radius 3.

M_3: Dotted disc around $z_0 = -3 - 2i$ with radius 3, excluding boundaries.

THANK YOU