Mathematics Department
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SuSe 2024

Exam: Complex Functions

26.August 2024

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(Signature)

Task no.	Points	Evaluator
1		
2		
3		
4		

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Task 1) [5 points]

Let i be the imaginary unit and R the rectangle

$$R := \left\{ z \in \mathbb{C} \mid z = x + iy,\ x, y \in \mathbb{R},\ |x| \leq \frac{\ln(2)}{\pi},\ |y| \leq \frac{1}{2} \right\}.$$

Determine the image of R under the mapping

$$f : \mathbb{C} \to \mathbb{C}, \qquad f(z) := 2e^{i\frac{\pi}{4}} \cdot e^{\pi z}.$$

Make a sketch of the image or describe the image in words.

Solution to Task 1) [5 points]

Let $\tilde{f}(z) := e^{\pi z} = e^{\pi x + i\pi y} = e^{\pi x} \cdot e^{i\pi y} =: \tilde{\rho} \cdot e^{i\tilde{\alpha}}$

$$\tilde{\rho} = \left|\tilde{f}(z)\right| = e^{\pi x} \in \left[e^{-\pi \frac{\ln(2)}{\pi}}, e^{\pi \frac{\ln(2)}{\pi}} \right] = \left[\frac{1}{e^{\ln 2}}, e^{\ln 2} \right] = \left[\frac{1}{2}, 2 \right].$$

$$\tilde{\alpha} = \arg\left(\tilde{f}(z)\right) = \pi y \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

[2 points]

$$f(z) = 2e^{i\frac{\pi}{4}} \tilde{f}(z)$$

$$\Longrightarrow |f(z)| = 2 \cdot |\tilde{f}(z)| \in [1, 4] \quad \text{and} \quad \arg\left(f(z)\right) = \frac{\pi}{4} + \arg\left(\tilde{f}(z)\right) \in \left[-\frac{\pi}{4}, \frac{3\pi}{4} \right].$$

Therefore,

$$f(R) = \left\{ w \in \mathbb{C} : 1 \leq |w| \leq 4, -\frac{\pi}{4} \leq \arg\left(w\right) \leq \frac{3\pi}{4} \right\}.$$

[2 points]

Sketch or Description:

$f(R)$ is an annular sector around the origin with an inner radius of 1 and an outer radius of 4, spanning from the bisector of the fourth quadrant to the bisector of the second quadrant.

[1 point]

Task 2) [6 points]

a) Determine a Möbius transform

$$T : \mathbb{C}^* \to \mathbb{C}^*, \quad T(z) := \frac{az+b}{cz+d}$$

that satisfies

$$T(3) = 0, \qquad T(0) = -6, \qquad T(-1) = \infty.$$

b) Which generalized circles in $\mathbb{C}$ are mapped onto straight lines by T?

c) Determine the images of the following generalized circles under T from part a):

(i) $K :=$ real axis,

(ii) $\hat{K} := \{z \in \mathbb{C} \mid |z-1| = 2\}$,

(iii) $\tilde{K} :=$ imaginary axis.

Solution for 2) [6 points]

a)

$$T(3) = 0,\, T(-1) = \infty \Longleftrightarrow T(z) = \frac{a(z-3)}{z+1}.$$

$$T(0) = -6 \Longrightarrow T(z) = \frac{2z-6}{z+1}.$$

[1 point]

b) A generalized circle is mapped onto a straight line if and only if the point -1 is located on that generalized circle.

[1 point]

c) (i) $K = \mathbb{R}$

Due to the real coefficients and the given images of -1, 0, 3, we have $T(\mathbb{R}) = \mathbb{R}$.
Alternative solution:

$$-1 \in \mathbb{R} \Longleftrightarrow T(\mathbb{R}) \text{ is a straight line.}$$

$$T(0) = -6,\, T(3) = 0 \Longleftrightarrow T(\mathbb{R}) = \mathbb{R}.$$

[1 point]

(ii) $\hat{K} := \{z \in \mathbb{C} \mid |z-1| = 2\}$

$$-1 \in \hat{K} \Longleftrightarrow T(\hat{K}) = \hat{g} \text{ is a straight line.}$$

Due to the symmetry of $\mathbb{R}$ and $\hat{K}$, we have $\hat{g} \perp T(\mathbb{R}) = \mathbb{R}$.

$$3 \in \hat{K} \Longrightarrow T(3) = 0 \in \hat{g}.$$

Therefore, $\hat{g} = i\mathbb{R}$, the imaginary axis.

[1 point]

(iii) $\tilde{K} :=$ imaginary axis.

$$-1 \notin i\mathbb{R} \Longleftrightarrow T(i\mathbb{R}) \text{ is a genuine circle } K_{i\mathbb{R}}.$$

$i\mathbb{R}$ symmetric to $\mathbb{R}$ $\Longrightarrow$ $T(i\mathbb{R})$ is symmetric to $T(\mathbb{R}) = \mathbb{R}$. The center of the image circle thus lies on the real axis.

Since $T(0) = -6$ and $T(\infty) = 2$, the center of the image circle is $M = -2$ and the radius $R = 4$.

[2 points]

Task 3) [6 points]

Given

$$f(z) = \frac{z+1}{z^3 + 3z^2}.$$

a) Determine and classify all isolated singularities of f.

b) Compute the residues of f at all isolated singularities.

c) How many different Laurent series of f exist for the expansion point $z_0 = 1$? Specify the rings in which the Laurent series converge to f.

d) Compute $\oint_{C_k} f(z)\,dz$ for $k = 1, 2$.

(i) $C_1 : [0, 2\pi] \to \mathbb{C}$, $C_1(t) = 3 + 2e^{it}$,

(ii) $C_2 : [0, 2\pi] \to \mathbb{C}$, $C_2(t) = 2e^{it}$.

Solution to Task 3)

a) $z^3 + 3z^2 = 0 \Longleftrightarrow z \in \{0, -3\}$.

There is a double zero of the denominator at $z_1 = 0$ and a simple zero of the denominator at $z_2 = -3$. The numerator does not vanish at any of these points. Thus, there is a pole of order 2 at $z_1 = 0$ and a pole of order 1 at $z_2 = -3$.

[**1 point**]

b) [**2 points**]

$$\operatorname{Res}(f, -3) = \left.\frac{z+1}{z^2}\right|_{z=-3} = \frac{-2}{9}.$$

$$\operatorname{Res}(f, 0) = \left(\frac{z+1}{z+3}\right)'_{z=0} = \left.\frac{z+3-z-1}{(z+3)^2}\right|_{z=0} = \frac{2}{9}.$$

c) [**1 point**]

There are three Laurent series, in the rings:

$$R_1 : |z-1| < 1, \quad R_2 : 1 < |z-1| < 4, \quad R_3 : 4 < |z-1|.$$

d) (i) $\oint_{C_1} f(z)\,dz = 0$ (Cauchy's Integral Theorem) [**1 point**]

(ii) $\oint_{C_2} f(z)\,dz = 2\pi i \operatorname{Res}(f, 0) = \dfrac{4\pi i}{9}$. [**1 point**]

Task 4) [3 points]

Given the function

$$f : \mathbb{C} \setminus \{1\} \to \mathbb{C}, \quad f(z) := \frac{z}{1 - \bar{z}}$$

and the curve

$$c : [0, \frac{\pi}{2}] \to \mathbb{C}, \quad c(t) = 1 + 2e^{it}.$$

Compute the curve integral

$$I_C := \int_c f(z)\, dz.$$

Solution to Task 4) [3 points]

$\dot{c}(t) = 2ie^{it}, \quad f(c(t)) = \frac{1+2e^{it}}{1-1-2e^{-it}} = -\frac{1}{2}(e^{it} + 2e^{2it}).$

$$I_C = \int_0^{\frac{\pi}{2}} (-\frac{1}{2}(e^{it} + 2e^{2it})) \cdot 2ie^{it} dt = -\int_0^{\frac{\pi}{2}} i(e^{2it} + 2e^{3it}))dt$$

$$= -\left[\frac{1}{2}e^{2it} + \frac{2}{3}e^{3it}\right]_0^{\frac{\pi}{2}} = -\frac{1}{2}(e^{i\pi} - e^0) + \frac{2}{3}(e^{i\frac{3\pi}{2}} - e^0)$$

$$= 1 - \frac{2}{3}(-i - 1) = \frac{5}{3} + i\frac{2}{3}.$$