Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Exercise class 7

Exercise 1:

For the following functions

a) $f(z) = \dfrac{z^2 + z - 2}{z^3 - 2z^2}$,

b) $f(z) = \dfrac{1 + z - \exp(z)}{z^4}$,

c) $f(z) = \cosh \dfrac{1}{z} - \sinh \dfrac{1}{z}$,

d) $f(z) = \dfrac{z - \pi}{\sin z}$

one determine:

position and type of the (finite) singularities, the corresponding residuals and the first four (non-vanishing) addends of the Laurent series around $z = 0$, converging for large z.

Solution:

a) The singularities of

$$f(z) = \frac{z^2+z-2}{z^3-2z^2} = \frac{(z+2)(z-1)}{z^2(z-2)} = \frac{1}{z^2} + \frac{1}{z-2}$$

are given by the denominator zeros

$$z_1 = 2\,, \quad z_2 = 0\,.$$

Since z_k are not zeros of the numerator, $z_1 = 2$ is a pole of order 1 and $z_2 = 0$ is a pole of order 2.

$$\text{Res}\,(f; z_1) = \left.\frac{z^2+z-2}{(z^3-2z^2)'}\right|_{z=z_1} = \left.\frac{z^2+z-2}{3z^2-4z}\right|_{z=2} = 1\,,$$

$$\begin{aligned}\text{Res}\,(f; z_2) &= \frac{1}{1!}\left.\left(z^2\left(\frac{1}{z^2}+\frac{1}{z-2}\right)\right)'\right|_{z=0} = \left.\left(1+\frac{z^2}{z-2}\right)'\right|_{z=0} \\ &= \left.\frac{2z(z-2)-z^2}{(z-2)^2}\right|_{z=0} = 0\end{aligned}$$

The Laurent development in the outer domain $|z| > 2$ results from:

$$\begin{aligned}f(z) &= \frac{1}{z^2} + \frac{1}{z-2} = \frac{1}{z^2} + \frac{1}{z}\cdot\frac{1}{1-2/z} \\ &= \frac{1}{z^2} + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \frac{1}{z^2} + \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right) \\ &= \frac{1}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \cdots\end{aligned}$$

b)

$$\begin{aligned}f(z) &= \frac{1+z-\exp(z)}{z^4} = \frac{1}{z^4}\left(1+z-\sum_{n=0}^{\infty}\frac{z^n}{n!}\right) \\ &= -\frac{1}{2!}\cdot\frac{1}{z^2} - \frac{1}{3!}\cdot\frac{1}{z} - \frac{1}{4!} - \frac{z}{5!} - \cdots\end{aligned}$$

Thus the only singularity $z_0 = 0$ is a pole of second order with

$$\text{Res}\,(f; z_0) = a_{-1} = -\frac{1}{3!}\,.$$

c)

$$\begin{aligned}f(z) &= \cosh\frac{1}{z} - \sinh\frac{1}{z} \\ &= \frac{1}{2}\left(\exp\left(\frac{1}{z}\right)+\exp\left(-\frac{1}{z}\right)\right) - \frac{1}{2}\left(\exp\left(\frac{1}{z}\right)-\exp\left(-\frac{1}{z}\right)\right) \\ &= \exp\left(-\frac{1}{z}\right) = \sum_{n=0}^{\infty}\underbrace{\frac{(-1)^n}{n!}}_{=:a_n}\frac{1}{z^n} = 1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{6z^3} + \cdots\end{aligned}$$

Thus the only singularity $z_0 = 0$ is essential with

$$\text{Res}\,(f; z_0) = a_{-1} = -1\,.$$

d) The singularities of $f(z) = \dfrac{z-\pi}{\sin z}$ result from:

$$\begin{aligned} 0 = \sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\frac{i}{2}\left(e^{-y+ix} - e^{y-ix}\right) \\ &= \frac{1}{2}\left(-ie^{-y}(\cos x + i\sin x) + ie^{y}(\cos x - i\sin x)\right) \\ &= \frac{1}{2}\left(\sin x(e^{y} + e^{-y}) + i\cos x(-e^{-y} + e^{y})\right) \\ &= \sin x\cosh y + i\cos x\sinh y \end{aligned}$$

All solutions are given by $x = k\pi$ and $y = 0$, thus by the already known real roots $z_k = k\pi$.

Such denominator zeros are simple, since it holds

$$(\sin)'(k\pi) = \cos k\pi = (-1)^k \neq 0.$$

The only (simple) numerator zero is $z_1 = \pi$. Hence z_1 is a removable singularity and all other $z_{k\neq 1}$ are first order poles. For the residuals we get

$$\text{Res}\,(f; z_1) = 0 \quad \text{and} \quad \text{Res}\,(f; z_{k\neq 1}) = \frac{k\pi - \pi}{(\sin)'(k\pi)} = \frac{(k-1)\pi}{(-1)^k}\,.$$

A Laurent series converging for all z with $|z| > R$ exists, because the singularities accumulate at infinity.

Exercise 2:

Let the function

$$f(z) = \frac{32}{z^4 + 4z^3 + 8z^2 + 16z + 16}$$

be given.

a) Determine the partial fraction decomposition of f with the help of Laurent series expansion.

b) Compute with the help of the residue theorem the integral

$$\oint_c f(z)\, dz$$

for the circumference $c: \; |z + 2 - 2i| = 3$.

Solution:

a) From the factorization

$$z^4 + 4z^3 + 8z^2 + 16z + 16 = (z^2 + 4)(z + 2)^2 = (z + 2i)(z - 2i)(z + 2)^2$$

the numerator roots are

$$z_0 = -2i\,, \quad z_1 = 2i\,, \quad z_2 = -2\,.$$

Thus z_0 and z_1 are poles of order 1 and z_2 is a pole of order 2. The principal part of the Laurent development at z_k , $k = 0, 1$ takes the form

$$h(z, z_k) = \frac{a_{-1,k}}{z - z_k}\,, \quad \text{where} \quad a_{-1,k} = \text{Res}(f(z); z_k)$$

For $z_0 = -2i$ it results

$$\begin{aligned} \text{Res}(f(z); -2i) &= \left.\frac{32}{(z - 2i)(z + 2)^2}\right|_{z=-2i} = \frac{32}{-4i(-2i + 2)^2} \\ &= \frac{32}{4i \cdot 8i} = -1 \end{aligned}$$

The same result is obtained by the Taylor series expansion of the holomorphic part of f :

$$\begin{aligned} f(z) &= \frac{1}{z + 2i} \cdot \underbrace{\frac{32}{(z - 2i)(z + 2)^2}}_{= g_1(z),\,(\text{holom.})} \\ &= \frac{1}{z + 2i}(g_1(-2i) + g_1'(-2i)(z + 2i) + \cdots) \end{aligned}$$

with $g_1(-2i) = \text{Res}(f(z); -2i) = -1$. Thus altogether one obtains

$$f(z) = \underbrace{-\frac{1}{z+2i}}_{= h(z,-2i)} + \underbrace{g_1'(-2i) + \cdots}_{\text{secondary part}}$$

For $z_1 = 2i$ it results accordingly

$$\begin{aligned} f(z) &= \frac{1}{z-2i} \cdot \underbrace{\frac{32}{(z+2i)(z+2)^2}}_{= g_2(z), \text{(holom.)}} \\ &= \frac{1}{z-2i}(g_2(2i) + g_2'(2i)(z-2i) + \cdots) \end{aligned}$$

with $g_2(2i) = \text{Res}(f(z); 2i) = -1$. Thus altogether one gets

$$f(z) = \underbrace{-\frac{1}{z-2i}}_{= h(z,2i)} + \underbrace{g_2'(2i) + \cdots}_{\text{secondary part}}$$

For the second order pole $z_2 = -2$ one gets the principal part of the Laurent series around z_2 via the Taylor series expansion of the holomorphic part of f :

$$\begin{aligned} f(z) &= \frac{1}{(z+2)^2} \cdot \underbrace{\frac{32}{z^2+4}}_{= g_3(z), \text{(holom.)}} \\ &= \frac{1}{(z+2)^2}\left(g_3(-2) + g_3'(-2)(z+2) + \frac{1}{2}g_3''(-2)(z+2)^2 + \cdots\right) \end{aligned}$$

After a short computation we get

$$g_3(-2) = 4\,, \quad g_3'(-2) = 2 \;=\; (\text{Res}(f(z); -2))$$

$$\Rightarrow \quad f(z) = \underbrace{\frac{4}{(z+2)^2} + \frac{2}{z+2}}_{= h(z,-2)} + \underbrace{g_3''(-2)/2 + \cdots}_{\text{secondary part}}$$

Therefore the complex partial fraction decomposition reads:

$$f(z) = h(z,-2i) + h(z,2i) + h(z,-2) = -\frac{1}{z+2i} - \frac{1}{z-2i} + \frac{4}{(z+2)^2} + \frac{2}{z+2}\,.$$

As real partial fraction decomposition it is:

$$f(z) = -\frac{2z}{z^2+4} + \frac{4}{(z+2)^2} + \frac{2}{z+2}\,.$$

b) Among the singularities of f

$$z_0 = -2i\,, \quad z_1 = 2i\,, \quad z_2 = -2$$

only z_1 and z_2 lie in the interior of c. From this applying the residue theorem it holds

$$\oint_c \frac{32}{z^4 + 4z^3 + 8z^2 + 16z + 16}\, dz = 2\pi i\,(\mathrm{Res}(f; 2i) + \mathrm{Res}(f; -2)) = 2\pi i$$

Dates of classes: 3.7.-7.7.