Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Homework 7

Exercise 1:

Determine the Laurent the Laurent development of the following functions and specify in each case the coefficients a_{-1} of the series:

a) $f(z) = \dfrac{\exp(z-2)}{z-2}$ at point $z_0 = 2\,,$

b) $f(z) = z^2 \cosh\left(\dfrac{1}{z+1}\right)$ at point $z_0 = -1\,,$

c) $f(z) = \dfrac{z - \sin z}{z^7}$ at point $z_0 = 0\,.$

Solution:

a) For $|z-2| > 0$ it holds:

$$\begin{aligned} f(z) &= \frac{\exp(z-2)}{z-2} = \frac{1}{z-2}\left(1 + \frac{z-2}{1!} + \frac{(z-2)^2}{2!} + \frac{(z-2)^3}{3!} + \cdots\right) \\ &= \frac{1}{z-2} + 1 + \frac{z-2}{2!} + \frac{(z-2)^2}{3!} + \frac{(z-2)^3}{4!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(z-2)^{n-1}}{n!} \qquad \Rightarrow \qquad a_{-1} = 1 \end{aligned}$$

b)
$$\begin{aligned}
f(z) &= z^2\cosh\left(\frac{1}{z+1}\right) = ((z+1)-1)^2\cosh\left(\frac{1}{z+1}\right)\\
&= ((z+1)^2-2(z+1)+1)\left(1+\frac{1}{2!(z+1)^2}+\frac{1}{4!(z+1)^4}+\cdots\right)\\
&= (z+1)^2-2(z+1)+\frac{3}{2}-\frac{1}{z+1}+\left(\frac{1}{2!}+\frac{1}{4!}\right)\frac{1}{(z+1)^2}\\
&\quad -\frac{2}{4!(z+1)^3}+\left(\frac{1}{4!}+\frac{1}{6!}\right)\frac{1}{(z+1)^4}\cdots\\
&= (z+1)^2-2(z+1)+\frac{3}{2}-\sum_{k=1}^{\infty}\frac{2}{(2k)!}(z+1)^{1-2k}\\
&\quad +\sum_{k=1}^{\infty}\left(\frac{1}{(2k)!}+\frac{1}{(2(k+1))!}\right)(z+1)^{-2k} \quad\Rightarrow\quad a_{-1}=-1
\end{aligned}$$

c) The only singularity of f is at the development point $z_0=0$. For $|z|>0$ it holds:

$$\begin{aligned}
f(z) &= \frac{z-\sin z}{z^7} = \frac{1}{z^7}\left(z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\frac{z^9}{9!}\mp\cdots\right)\right)\\
&= \frac{1}{3!}\cdot\frac{1}{z^4}-\frac{1}{5!}\cdot\frac{1}{z^2}+\frac{1}{7!}-\frac{z^2}{9!}\pm\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}z^{2n-6}\\
&\Rightarrow\quad a_{-1}=0
\end{aligned}$$

Exercise 2:

Compute the following integrals using the residual calculus

a) $$\int_0^\infty \frac{1}{x^{5/2}+13x^{3/2}+36x^{1/2}}\,dx\,,$$

b) $$\int_0^{2\pi} \frac{1}{2+\cos x}\,dx\,,$$

c) $$\int_{-\infty}^\infty \frac{1}{x^4+10x^2+9}\,dx\,,$$

d) $$\int_{-\infty}^\infty \frac{\cos(3x)}{x^2-6x+10}\,dx\,.$$

Solution:

a)
$$\begin{aligned}
&\int_0^\infty \frac{1}{x^{5/2}+13x^{3/2}+36x^{1/2}}\,dx \\
&= \int_0^\infty \frac{dx}{x^{1/2}(x^2+13x+36)} = \int_0^\infty \frac{dx}{x^{1/2}(x+4)(x+9)} \\
&= \frac{2\pi i}{1-e^{-2\pi i/2}}\left(\operatorname{Res}\left(\frac{1}{x^{1/2}(x+4)(x+9)};-4\right)+\operatorname{Res}\left(\frac{1}{x^{1/2}(x+4)(x+9)};-9\right)\right) \\
&= \pi i\left(\frac{1}{\sqrt{-4}(-4+9)}+\frac{1}{\sqrt{-9}(-9+4)}\right) = \frac{\pi}{5}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\pi}{30}
\end{aligned}$$

b)
$$\begin{aligned}
\int_0^{2\pi} \frac{1}{2+\cos x}\,dx &= \oint_{|z|=1} \frac{1}{2+(z+1/z)/2}\,\frac{dz}{iz} = \frac{2}{i}\oint_{|z|=1}\frac{dz}{z^2+4z+1} \\
&= \frac{2}{i}\oint_{|z|=1} \underbrace{\frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}}_{=f(z)}\,dz \\
&= 2\pi i\cdot\frac{2}{i}\cdot\operatorname{Res}\left(f;-2+\sqrt{3}\right) \\
&= 4\pi\cdot\frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{2\pi}{\sqrt{3}}
\end{aligned}$$

c)
$$\int_{-\infty}^{\infty} \frac{1}{x^4+10x^2+9}\,dx = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)}$$
$$= \int_{-\infty}^{\infty} \underbrace{\frac{1}{(x+i)(x-i)(x+3i)(x-3i)}}_{=f(x)}\,dx \;=\; 2\pi i\left(\mathrm{Res}(f;i)+\mathrm{Res}(f;3i)\right)$$
$$= 2\pi i\left(\frac{1}{(i+i)(i^2+9)} + \frac{1}{((3i)^2+1)(3i+3i)}\right) \;=\; \pi\left(\frac{1}{8}+\frac{1}{(-8)\cdot 3}\right) \;=\; \frac{\pi}{12}$$

d)
$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2-6x+10}\,dx \;=\; \int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{e^{3ix}}{(x-3)^2+1}\right)dx$$
$$= \mathrm{Re}\left(\int_{-\infty}^{\infty} \underbrace{\frac{e^{3ix}}{(x-(3+i))(x-(3-i))}}_{=f(x)}\,dx\right)$$
$$= \mathrm{Re}\left(2\pi i\ \mathrm{Res}(f;3+i)\right) \;=\; \mathrm{Re}\left(2\pi i\,\frac{e^{3i(3+i)}}{3+i-(3-i)}\right)$$
$$= \mathrm{Re}\left(\pi e^{-3+9i}\right) = \pi e^{-3}\cos 9$$

Hand in until: 7.7.