Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Exercise class 6

Exercise 1:

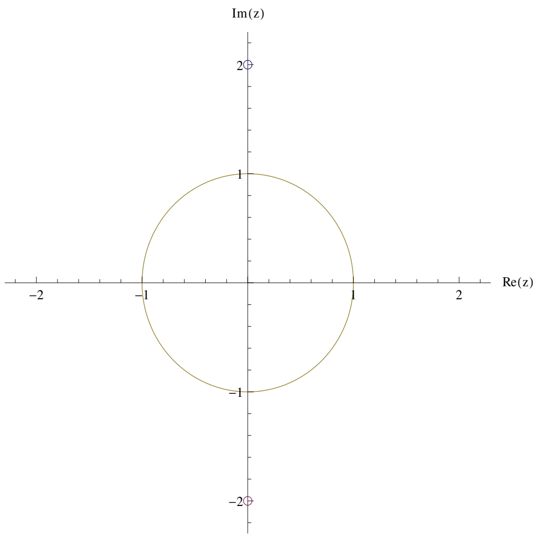
Compute, if possible with the help of Cauchy's integral formula, the following curve integrals (let all occurring curves be positively oriented):

a) $\displaystyle\oint\limits_{|z|=1} \frac{1}{z^2+4}\,dz\,,$ b) $\displaystyle\oint\limits_{|z|=2} \frac{z^2+1}{z+1}\,dz\,,$ c) $\displaystyle\oint\limits_{|z+i|=1} \frac{\cos z}{(z+i)^2}\,dz\,,$

d) $\displaystyle\oint\limits_{|z-2|=1} \sin z + \frac{\ln z}{(z-2)^2}\,dz\,,$ e) $\displaystyle\oint\limits_{|z+i|=2} \frac{\cos z}{z^3}\,dz\,,$ f) $\displaystyle\oint\limits_{|z|=4} \frac{\cosh z}{(z-i\pi)^5}\,dz\,.$

Solution:

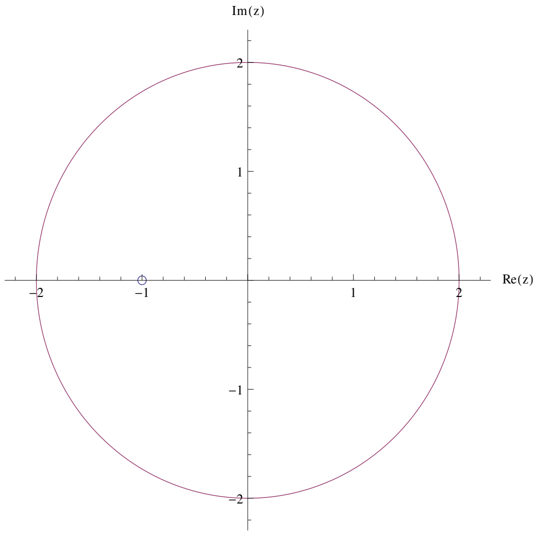
a) The singularities $z_{1,2} = \pm 2i$ do not lie on the circumference $|z| = 1$.



Cauchy's integral theorem:

$$\oint\limits_{|z|=1} \frac{1}{z^2+4}\,dz = 0$$

b)

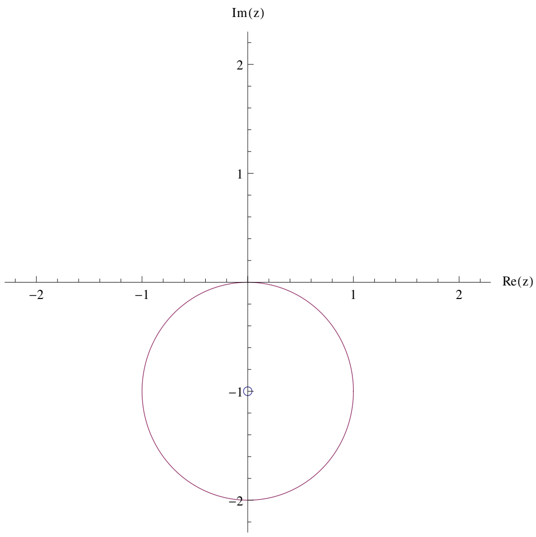


The singularity $z_1 = -1$ lies on the circumference $|z| = 2$.

Cauchy's integral formula:

$$\oint\limits_{|z|=2} \frac{z^2+1}{z+1}\,dz \;= 2\pi i((-1)^2+1) = 4\pi i$$

c)



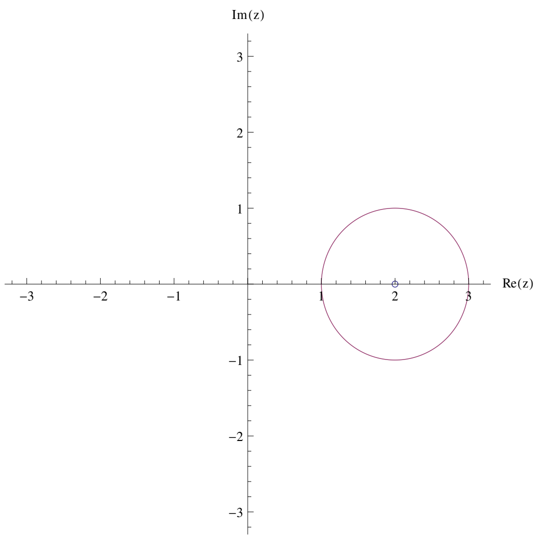
The singularity $z_1 = -i$ lies on the circumference $|z+i| = 1$.

Generalized Cauchy's integral formula:

$$\oint\limits_{|z+i|=1} \frac{\cos z}{(z+i)^2}\,dz = \frac{2\pi i}{1!}\,(\cos z)'\,|_{z=-i}$$

$$= 2\pi i \sin i = 2\pi i \tfrac{e^{i^2}-e^{-i^2}}{2i} = \pi\left(e^{-1}-e\right)$$

d)



The singularity $z_1 = 2$ of the second addend in the integrand lies in the circumference $|z-2| = 1$.

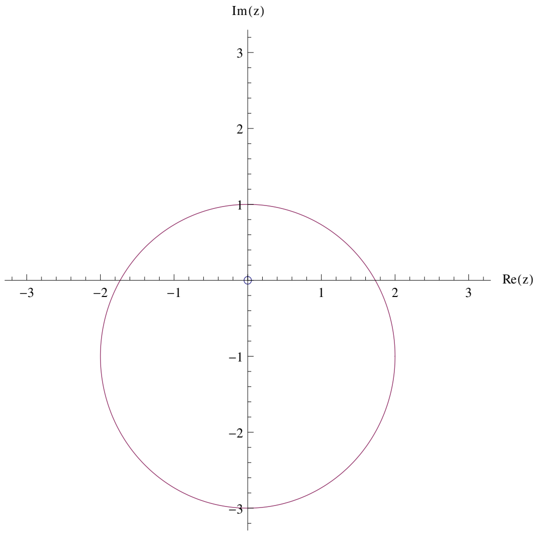
Cauchy's integral theorem and generalized Cauchy's integral formula:

$$\oint\limits_{|z-2|=1} \sin z + \frac{\ln z}{(z-2)^2}\,dz$$

$$= \oint\limits_{|z-2|=1} \frac{\ln z}{(z-2)^2}\,dz$$

$$= 2\pi i \ln' z|_{z=2} = \pi i$$

e)

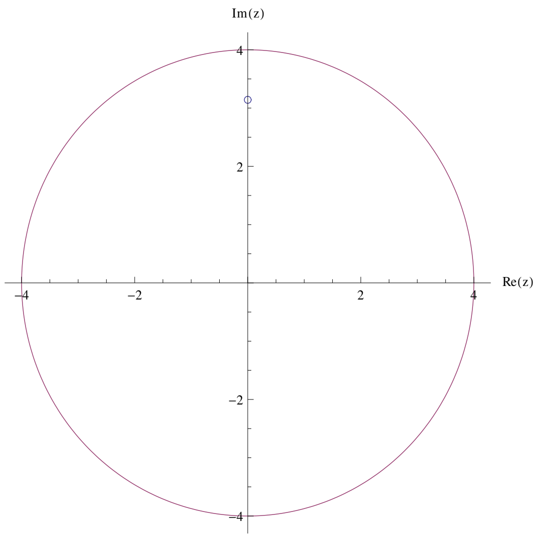


The singularity $z_1 = 0$ lies on the circumference $|z+i| = 2$.

Generalized Cauchy's integral formula:

$$\oint\limits_{|z+i|=2} \frac{\cos z}{z^3}\, dz \;=\; 2\pi i \frac{(\cos z)''}{2!}\bigg|_{z=0} = -\pi i$$

f) The singularity $z_1 = i\pi$ lies on the circumference $|z| = 4$.



Generalized Cauchy's integral formula:

$$\oint\limits_{|z|=4} \frac{\cosh z}{(z-i\pi)^5}\, dz$$

$$= \frac{2\pi i\, (\cosh z)''''|_{z=i\pi}}{4!}$$

$$= \frac{\pi i \cosh i\pi}{12} = \frac{\pi i \cos \pi}{12}$$

$$= -\frac{\pi i}{12}$$

Exercise 2:

Indicate **all** the power series expansions of the function

$$f(z) = \frac{5z}{z^2 + z - 6}$$

at the development point $z_0 = i$. Where do the series converge in each case?

Solution:

Factorization of the denominator $z^2 + z - 6 = (z-2)(z+3)$ yields the singularities of the function at $z_1 = 2$ and $z_2 = -3$. A partial fraction decomposition returns:

$$f(z) = \frac{5z}{z^2 + z - 6} = \frac{3}{z+3} + \frac{2}{z-2}\,.$$

In view of the position of the development point at $z_0 = i$ and of the two singularities $z_1 = 2$ and $z_2 = -3$, one can deduce that there will be a Taylor series expansion in the disk $|z - i| < \sqrt{5}$, a Laurent series expansion in the circular ring $\sqrt{5} < |z-i| < \sqrt{10}$ and a different Laurent series expansion in the external space $\sqrt{10} < |z - i|$.

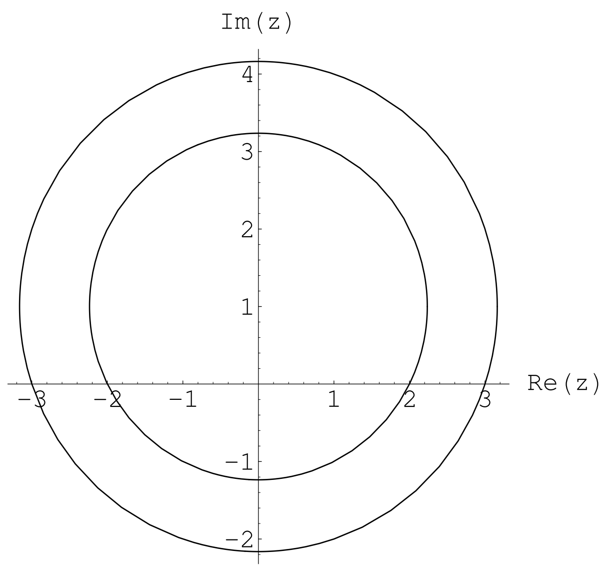


Bild 2: Convergence domains of the Laurent series expansion around $z_0 = i$

With the help of the sum formula of the geometric series in the corresponding convergence domains, the partial fractions can be represented by power series expansions:

$$|z-i| < \sqrt{5} \quad : \quad \frac{2}{z-2} = \frac{2}{-2+i+(z-i)} = \frac{2}{-2+i} \cdot \frac{1}{1-(z-i)/(2-i)}$$
$$= \frac{2}{-2+i} \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^n} = \sum_{n=0}^{\infty} \frac{-2}{(2-i)^{n+1}} (z-i)^n$$

$$|z-i| > \sqrt{5} \quad : \quad \frac{2}{z-2} = \frac{2}{-2+i+(z-i)} = \frac{2}{z-i} \cdot \frac{1}{1-(2-i)/(z-i)}$$
$$= \frac{2}{z-i} \sum_{n=0}^{\infty} \frac{(2-i)^n}{(z-i)^n} = \sum_{n=-\infty}^{-1} \frac{2}{(2-i)^{n+1}} (z-i)^n$$

$$|z-i| < \sqrt{10} \quad : \quad \frac{3}{z+3} = \frac{3}{3+i+(z-i)} = \frac{3}{3+i} \cdot \frac{1}{1+(z-i)/(3+i)}$$
$$= \frac{3}{3+i} \sum_{n=0}^{\infty} \frac{(-1)^n}{(3+i)^n} (z-i)^n = \sum_{n=0}^{\infty} \frac{-3}{(-3-i)^{n+1}} (z-i)^n$$

$$|z-i| > \sqrt{10} \quad : \quad \frac{3}{z+3} = \frac{3}{3+i+(z-i)} = \frac{3}{z-i} \cdot \frac{1}{1+(3+i)/(z-i)}$$
$$= \frac{3}{z-i} \sum_{n=0}^{\infty} \frac{(-1)^n (3+i)^n}{(z-i)^n} = \sum_{n=-\infty}^{-1} \frac{3}{(-3-i)^{n+1}} (z-i)^n$$

Taylor series with convergence in the disk $|z-i| < \sqrt{5}$:

$$f(z) = \frac{3}{z+3} + \frac{2}{z-2} = \underbrace{\sum_{n=0}^{\infty} \left(\frac{-2}{(2-i)^{n+1}} + \frac{-3}{(-3-i)^{n+1}} \right) (z-i)^n}_{\text{Secondary part}} .$$

Laurent series with convergence in the circular ring $\sqrt{5} < |z-i| < \sqrt{10}$:

$$f(z) = \frac{3}{z+3} + \frac{2}{z-2} = \underbrace{\sum_{n=-\infty}^{-1} \frac{2}{(2-i)^{n+1}} (z-i)^n}_{\text{Principal part}} + \underbrace{\sum_{n=0}^{\infty} \frac{-3}{(-3-i)^{n+1}} (z-i)^n}_{\text{Secondary part}} .$$

Laurent series with convergence in the outer ring $\sqrt{10} < |z-i|$:

$$f(z) = \frac{3}{z+3} + \frac{2}{z-2} = \underbrace{\sum_{n=-\infty}^{-1} \left(\frac{2}{(2-i)^{n+1}} + \frac{3}{(-3-i)^{n+1}} \right) (z-i)^n}_{\text{Principal part}} .$$

Dates of classes: 19.6.- 23.6.