Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Homework 6

Exercise 1:

Compute directly and with the help of a primitive function

a) $\displaystyle\int\limits_c 2z - 3\,dz$ along the segment from $-1-i$ to $-i$,

b) $\displaystyle\int\limits_c z\cosh z\,dz$ for $c(t) = it$ with $0 \le t \le 1$,

c) $\displaystyle\int\limits_{-i}^{1} \frac{z+1}{z}\,dz$ for $c(\varphi) = e^{i\varphi}$ (positively oriented),

d) $\displaystyle\int\limits_{-i}^{i} \sin z\,dz$ for $c(t) = it$, $-1 \le t \le 1$.

Solution:

a) Direct:

Curve parametrisation:

$c(t) = -1 - i + t\,, \quad 0 \le t \le 1 \ \Rightarrow\ \dot c(t) = 1$

$$\begin{aligned}\int\limits_c 2z - 3\,dz &= \int\limits_0^1 (2c(t) - 3)\dot c(t)\,dt = \int\limits_0^1 2(-1-i+t) - 3\,dt \\ &= \left((-5-2i)t + t^2\right)\Big|_0^1 = -5 - 2i + 1 = -4 - 2i\end{aligned}$$

Primitive function:

$$\int_c 2z-3\,dz = \int_{-1-i}^{-i} 2z-3\,dz = \left(z^2-3z\right)\Big|_{-1-i}^{-i}$$
$$= (-i)^2-3(-i)-\left((-i-1)^2-3(-i-1)\right)$$
$$= -1+3i-(2i+3i+3) = -4-2i$$

b) Direct: $c(t)=it \Rightarrow \dot{c}(t)=i$ with $0\le t\le 1$ and $\cosh(it)=\cos t$

$$\int_c z\cosh z\,dz = \int_0^1 \dot{c}(t)c(t)\cosh c(t)\,dt = \int_0^1 (i)^2 t\cosh(it)\,dt$$
$$= -\int_0^1 t\cos t\,dt = -\,t\sin t\big|_0^1 + \int_0^1 \sin t\,dt$$
$$= -\,(t\sin t+\cos t)\big|_0^1 = -\sin 1-\cos 1+1$$

Primitive function: with $\sinh i = \frac{1}{2}(e^i-e^{-i}) = i\sin 1$

$$\int_c z\cosh z\,dz = \int_0^i z\cosh z\,dz = (z\sinh z-\cosh z)\big|_0^i$$
$$= i\sinh i-\cosh i+1 = -\sin 1-\cos 1+1$$

c) Direct:

$$\int_{-i}^{1} \frac{z+1}{z}\,dz = \int_{-\pi/2}^{0} \left(1+\frac{1}{c(\varphi)}\right)\dot{c}(\varphi)\,d\varphi = \int_{-\pi/2}^{0} \left(1+\frac{1}{e^{i\varphi}}\right) ie^{i\varphi}\,d\varphi$$
$$= i\int_{-\pi/2}^{0} 1+e^{i\varphi}\,d\varphi = i\left(\varphi+\frac{e^{i\varphi}}{i}\right)\Bigg|_{-\pi/2}^{0} = 1+i\left(1+\frac{\pi}{2}\right)$$

Primitive function:

In the slitted plane $\mathbb{C}\backslash\mathbb{R}_-^0$, $\ln z$ is the primitive function of $\frac{1}{z}$.

$$\int_{-i}^{1} \frac{z+1}{z}\,dz = \int_{-i}^{1} 1+\frac{1}{z}\,dz = (z+\ln z)\big|_{-i}^{1}$$
$$= 1+\ln|1|+i\cdot 0-\left(-i+\ln|-i|+i\frac{-\pi}{2}\right)$$
$$= 1+i\left(1+\frac{\pi}{2}\right)$$

d) Direct: $c(t) = it$, $\dot{c}(t) = i$ with $-1 \leq t \leq 1$

$$\begin{aligned}
\int_{-i}^{i} \sin z \, dz &= \int_{-1}^{1} \dot{c}(t) \sin(c(t)) \, dt = \int_{-1}^{1} i \sin(it)) \, dt \\
&= \int_{-1}^{1} i \frac{1}{2i} \left(e^{iit} - e^{-iit}\right) \, dt = -\frac{1}{2} \int_{-1}^{1} e^{t} - e^{-t} \, dt \\
&= -\frac{1}{2} \left(e^{t} + e^{-t}\right) \Big|_{-1}^{1} = 0
\end{aligned}$$

Primitive function:

$$\int_{-i}^{i} \sin z \, dz = -\cos z|_{-i}^{i} = -\frac{1}{2} \left(e^{ii} + e^{-ii} - e^{-ii} - e^{ii}\right) = 0$$

Exercise 2:

a) Compute the Taylor series of $f(z) = \displaystyle\int_0^z \frac{d\xi}{4+\xi^2}$ at development point $z_0 = 0$ and determine the convergence radius.

b) Determine the convergence radii of the following Taylor series functions at the given development points z_0 and without computing the series itself:

(i) $f(z) = \dfrac{3}{z^2+2z+5}\,, \quad z_0 = i$ and $z_0 = 0\,,$

(ii) $f(z) = \dfrac{2}{e^z-1}\,, \quad z_0 = 2\pi(1+i)\,,$

(iii) $f(z) = \dfrac{z}{\ln(3-2z)}\,, \quad z_0 = 0$ and $z_0 = \dfrac{11}{8}\,.$

Solution:

a) In the circular disk $|\xi| < 2 =: r$ with the help of the geometric series one gets

$$\frac{1}{4+\xi^2} = \frac{1}{4}\cdot\frac{1}{1+(\xi/2)^2} = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\left(\frac{\xi}{2}\right)^2\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\xi}{2}\right)^{2n}.$$

Since the series converges uniformly in the circular disk, it can be integrated member by member:

$$\begin{aligned} f(z) &= \int_0^z \frac{d\xi}{4+\xi^2} = \int_0^z \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\xi}{2}\right)^{2n} d\xi = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\int_0^z\left(\frac{\xi}{2}\right)^{2n} d\xi \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\frac{2(-1)^n}{2n+1}\left(\frac{\xi}{2}\right)^{2n+1}\Bigg|_0^z = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{z}{2}\right)^{2n+1} \end{aligned}$$

b) (i)

$$f(z) = \frac{3}{z^2+2z+5} = \frac{3}{(z-(-1+2i))(z-(-1-2i))}$$

The singularities lie at $z_1 = -1+2i$ and $z_2 = -1-2i$. Thus, the convergence radius of the Taylor series at development point z_0 is obtained by

$$r = \min\{|z_1-z_0|, |z_2-z_0|\}\,.$$

$$z_0 = i: \quad r_1 = \min\{|-1+2i-i|, |-1-2i-i|\} = \min\{\sqrt{2}, \sqrt{10}\} = \sqrt{2}$$

$$z_0 = 0: \quad r_2 = \min\{|-1+2i|, |-1-2i|\} = \sqrt{5}$$

(ii) The singularities of $f(z) = \dfrac{2}{e^z - 1}$ result from

$0 = e^z - 1 \quad \Leftrightarrow \quad 1 = e^x \cos y + i e^x \sin y$
$\Rightarrow \quad e^x \sin y = 0 \quad \Rightarrow \quad y = k\pi$
$\Rightarrow \quad e^x \cos(k\pi) = e^x(-1)^k = 1$
$\Rightarrow \quad x = 0$ and $k = 2n$ with $n \in \mathbb{Z}$.

Thus the singularities lie at $\tilde{z}_n = 2n\pi i$. The radius of convergence for $z_0 = 2\pi(1+i)$ results from:

$$r = \min_n\{|\tilde{z}_n - z_0|\} = \min_n\{|2n\pi i - 2\pi - 2\pi i|\} = |-2\pi| = 2\pi$$

(iii) For the function $f(z) = \dfrac{z}{\ln(3-2z)}$ the following definition gaps hold:

First case:

The principal value of the logarithm $\ln(3-2z)$ is only defined in the slit plane, i.e. the real numbers x with

$$3 - 2x \le 0 \Leftrightarrow \frac{3}{2} \le x$$

are excluded.

Second case:

Zeroes of the denominator must be excluded:
$0 = \ln(3-2z) = \ln|3-2z| + i\arg(3-2z) \quad \Rightarrow$
$\arg(3-2z) = 0$, thus z is real and in light of the first case $|3-2z| = 1$ only returns $z = 1$.

The convergence radius is given by the smallest distance of the development point z_0 from the closest definition gap.

For $z_0 = \dfrac{11}{8}$ it results

$$r_1 = \min\left\{\left|\frac{3}{2} - \frac{11}{8}\right|, \left|1 - \frac{11}{8}\right|\right\} = \min\left\{\frac{1}{8}, \frac{3}{8}\right\} = \frac{1}{8}.$$

For $z_0 = 0$ it results $r_2 = \min\left\{\left|\dfrac{3}{2} - 0\right|, |1-0|\right\} = 1$.

Hand in until: 23.6.