Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Exercise class 5

Exercise 1:

For $f : \mathbb{C} \to \mathbb{C}$ mit $f(z) = z^3$ compute

a) $A := \frac{1}{2}\left(f_x(z_0) - i f_y(z_0)\right)$ and

b) $B := \frac{1}{2}\left(f_x(z_0) + i f_y(z_0)\right)$.

Compare the results with partial derivatives of f with respect to the independent variables z und $\bar{z}$, that is with

$$\frac{\partial f}{\partial z}\,, \quad \frac{\partial f}{\partial \bar{z}}\,.$$

For this apply formally the usual rules of derivation on the real field.

Solution:

f has the following decomposition into real and imaginary part

$$f(z_0) = z_0^3 = (x_0 + iy_0)^3 = x_0^3 - 3x_0y_0^2 + i(3x_0^2y_0 - y_0^3) = f(x_0, y_0)\,.$$

One gets

$$f_x(z_0) = f_x(x_0, y_0) = 3x_0^2 - 3y_0^2 + i6x_0y_0, \quad f_y(z_0) = f_y(x_0, y_0) = -6x_0y_0 + i(3x_0^2 - 3y_0^2)$$

a) $A = \frac{1}{2}\left(f_x(z_0) - i f_y(z_0)\right) = \frac{1}{2}\left(3x_0^2 - 3y_0^2 + i6x_0y_0 - i(-6x_0y_0 + i(3x_0^2 - 3y_0^2))\right)$
$= \frac{1}{2}\left(6x_0^2 - 6y_0^2 + i12x_0y_0\right) = 3\left(x_0^2 - y_0^2 + i2x_0y_0\right) = 3(x_0 + iy_0)^2 = 3z_0^2$
By purely formal differentiation as in the real numbers, one obtains:
$\frac{\partial f}{\partial z}(z_0) = 3z_0^2\,.$

b) $B = \frac{1}{2}\left(f_x(z_0) + i f_y(z_0)\right)$
$= \frac{1}{2}\left(3x_0^2 - 3y_0^2 + i6x_0y_0 + i(-6x_0y_0 + i(3x_0^2 - 3y_0^2))\right) = 0$
By purely formal differentiation as in the real numbers, one obtains:
$\frac{\partial f}{\partial \bar{z}}(z_0) = 0\,.$

Exercise 2:

a) Decide (with justifications) whether

(i) $f(z) = z^2 + \bar{z}^2 + 4i \cdot \mathrm{Re}(z)\mathrm{Im}(z) + i$ is holomorphic,

(ii) $g(z) = \mathrm{Re}(e^z)$ is holomorphic,

(iii) $\mathrm{Re}(z^{10} + \sin^7 z)$ is harmonic.

b) Let the function

$$v(x, y) = 2xy - 6y + e^x \sin y$$

be given.

(i) Show that v is harmonic.

(ii) For $v(x, y)$ determine a function $u(x, y)$ such that the function $f(z) = u(x, y) + iv(x, y)$ with $z = x + iy$ is holomorphic.

Solution:

a) (i) f is holomorphic since for $z = x + iy$ it holds

$$\begin{aligned} f(z) &= z^2 + \bar{z}^2 + 4i \cdot \mathrm{Re}(z)\mathrm{Im}(z) + i \\ &= x^2 - y^2 + i2xy + x^2 - y^2 - i2xy + i4xy + i \\ &= \underbrace{2(x^2 - y^2)}_{=u(x,y)} + i\underbrace{(4xy + 1)}_{=v(x,y)} . \end{aligned}$$

Thus u, v are continuously partially differentiable and the Cauchy-Riemann differential equations

$$u_x = 4x = v_y \ , \quad v_x = 4y = -u_y$$

hold.

(ii) $g(z) = \mathrm{Re}(e^z) = e^x \cos y$ is real-valued and non-constant, therefore it is not holomorphic.

(iii) $\mathrm{Re}(z^{10} + \sin^7 z)$ is harmonic since $z^{10} + \sin^7 z$ is holomorphic.

b) (i) $$\begin{aligned} \Delta v &= (2xy - 6y + e^x \sin y)_{xx} + (2xy - 6y + e^x \sin y)_{yy} \\ &= e^x \sin y + (-e^x \sin y) = 0 \end{aligned}$$

(ii) In order for $f(z) = u(x, y) + iv(x, y)$ to be holomorphic in $\mathbb{C}$, the Cauchy-Riemann differential equations must be satisfied:

$u_x = v_y = (2xy - 6y + e^x \sin y)_y = 2x - 6 + e^x \cos y$

$\Rightarrow \quad u = x^2 - 6x + e^x \cos y + c(y)$

$u_y = -e^x \sin y + c'(y) = -v_x = -(2xy - 6y + e^x \sin y)_x = -2y - e^x \sin y$

$\Rightarrow \quad c'(y) = -2y \quad \Rightarrow \quad c(y) = -y^2 + c \in \mathbb{R}$

Being $u(x, y) = x^2 - y^2 - 6x + e^x \cos y + c$ and v continuously partially differentiable, f is holomorphic in $\mathbb{C}$.

Remark:

For $f(z) = (z - 3)^2 + e^z$ and $c = 9$ it results $u(x, y) = \mathrm{Re} f$ and $v(x, y) = \mathrm{Im} f$.

Exercise 3:

Let the curves $c_1(t) = it$ and $c_2(t) = e^{it}$ be given, in each case for $0 < t < \pi$.

a) Draw the curves c_1 and c_2 in the z-plane and determine their intersection point with intersection angle.

b) Into which image curves of the w-plane are c_1 and c_2 mapped into by the principal value of $w = \ln z$? Check whether the intersection angle of the curves and the local length ratio are preserved.

Solution:

$w = \ln z = \ln|z| + i \arg z$ with $-\pi < \arg z < \pi$ (principal value)

a) $c_1(t) = it$ with $0 < t < \pi$:
segment between 0 and $i\pi$ without endpoints.

$c_2(t) = e^{it}$ with $0 < t < \pi$:
upper unit circle without endpoints 1 and -1 .

Intersection point: $c_1(1) = i = c_2\left(\frac{\pi}{2}\right)$

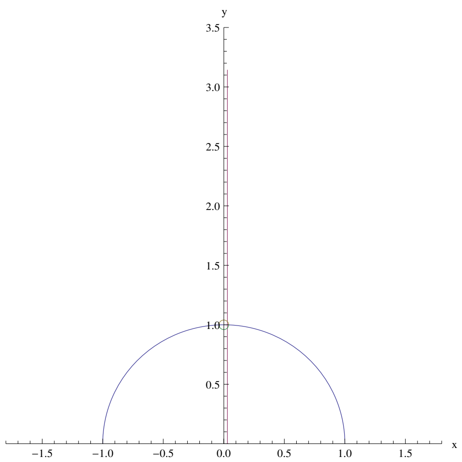

Figure 3 a): $c_1(t)$, $c_2(t)$ and intersection point $z_s = i$ in the z-plane

Derivatives of the curves at the intersections: $\dot{c}_1(t) = i \Rightarrow \dot{c}_1(1) = i$ and $\dot{c}_2(t) = ie^{it} \Rightarrow \dot{c}_2\left(\frac{\pi}{2}\right) = i^2 = -1$

Intersection angle:

$$\gamma = \angle\left(\dot{c}_2\left(\frac{\pi}{2}\right), \dot{c}_1(1)\right) = \arg \dot{c}_2\left(\frac{\pi}{2}\right) - \arg \dot{c}_1(1)$$
$$= \arg(-1) - \arg i = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

b) The image curves are denoted by d_1 and d_2.

$d_1(t) = \ln(c_1(t)) = \ln|it| + i\arg(it) = \ln t + \dfrac{i\pi}{2}$ with $0 < t < \pi$

(Parallel half-line in the real axis without endpoints)

$d_2(t) = \ln(c_2(t)) = \ln|e^{it}| + i\arg(e^{it}) = \ln 1 + it = it$ with $0 < t < \pi$

(Segment between 0 and $i\pi$ without endpoints)

Intersection point: $\quad d_1(1) = \dfrac{i\pi}{2} = d_2\left(\dfrac{\pi}{2}\right)$

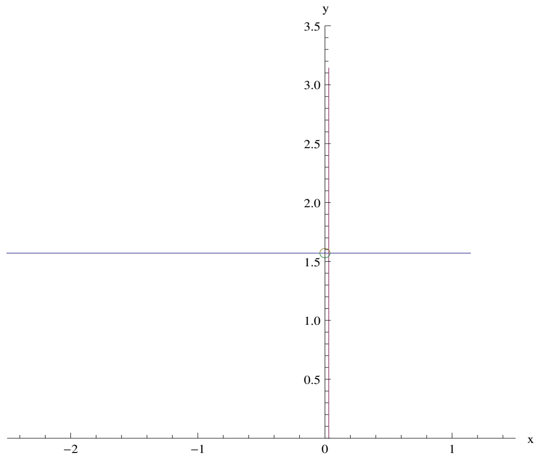

Figure 3 b): $d_1(t)$, $d_2(t)$ and intersection $w_s = \ln z_s = \dfrac{i\pi}{2}$ in the w-plane

Derivatives of the curves at the intersections:

$\dot{d}_1(t) = \dfrac{1}{t} \Rightarrow \dot{d}_1(1) = 1$ and $\dot{d}_2(t) = i \Rightarrow \dot{d}_2\left(\dfrac{\pi}{2}\right) = i$

Intersection angle:

$\tilde{\gamma} = \angle\left(\dot{d}_2\left(\dfrac{\pi}{2}\right), \dot{d}_1(1)\right) = \arg \dot{d}_2\left(\dfrac{\pi}{2}\right) - \arg \dot{d}_1(1)$

$= \arg i - \arg 1 = \dfrac{\pi}{2} - 0 = \dfrac{\pi}{2}$

local length ratio:

$$\frac{|\dot{c}_2\left(\frac{\pi}{2}\right)|}{|\dot{c}_1(1)|} = \frac{|-1|}{|i|} = 1 = \frac{|i|}{|1|} = \frac{|\dot{d}_2\left(\frac{\pi}{2}\right)|}{|\dot{d}_1(1)|}$$

Remark:

The stretch factor $f'(c(t))$ in

$$\dot{d}(t) = \frac{d}{dt}\left(f(c(t))\right) = f'(c(t))\dot{c}(t)$$

is shortened at the intersection point.

The intersection angle and the local length ratio are preserved because the curves run in the holomorphic domain of $\ln z$.

Dates of classes: 5.6. - 9.6.