Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Homework 5

Exercise 1:

a) Draw the line $G = \{z \in \mathbb{C} \mid z = -1 + it\ ,\ t \in \mathbb{R}\}$ and the circumference $K = \{z \in \mathbb{C} \mid |z-2| = \sqrt{5}\}$ and compute the two points z_1 and z_2 which lie symmetrically to G and K.

b) Determine all conformal functions

$$T(z) = \frac{az+b}{cz+d}$$

with $T(z_1) = 0$ and $T(z_2) = \infty$.

c) Draw the images of G and K under T, in case it holds still $T(-1) = -1$.

Solution:

a)

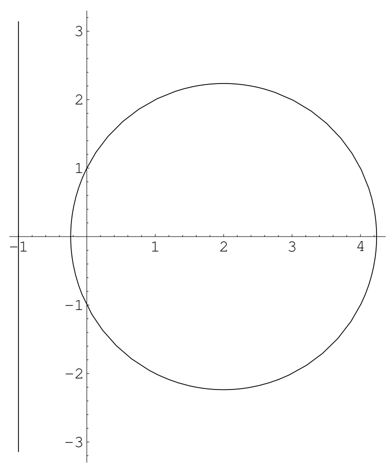

Figure 1 a): Line G and circumference K

Since z_1 and z_2 lie symmetrically with respect to the line G, the line H running through both points is perpendicular to G. Because of the symmetry of the points respect to K, such line H also runs through the centre $z_0 = 2$ of K. Thus H is the real axis and it holds $z_{1,2} \in \mathbb{R}$.

From the symmetry respect to G it follows $z_1 = -1 + a$ and $z_2 = -1 - a$ with $a \in \mathbb{R}$.

The symmetry to $K = \{z \in \mathbb{C} \mid |z - 2| = \sqrt{5}\}$ returns the constraint

$$\begin{aligned} 5 = (\sqrt{5})^2 &= (z_1 - z_0)(\bar{z}_2 - \bar{z}_0) \\ &= (-1 + a - 2)(-1 - a - 2) \\ &= -(a - 3)(a + 3) = -(a^2 - 9) \\ \Rightarrow a^2 &= 4 \Rightarrow a = \pm 2 \end{aligned}$$

$$\Rightarrow z_1 = -1 \pm 2 = 1 \vee -3 \quad , \quad z_2 = -1 \mp 2 = -3 \vee 1$$

b) With

$$T(z) = k \cdot \frac{z - z_1}{z - z_2}\,, \quad k \in \mathbb{C}\backslash\{0\}$$

it is $\quad w_1 = T(z_1) = 0$ and $w_2 = T_2(z_2) = \infty$.

T is a Möbius transformation being $k \neq 0$, thus $ad - bc = k(z_1 - z_2) \neq 0$.

T is holomorphic for $z \in \mathbb{C}\backslash\{z_2\}$ and therefore also conformal in this domain, since $T'(z) = \dfrac{ad - bc}{(z - z_2)^2} \neq 0$ holds.

c) Since z_1 and z_2 lie symmetrically to G and K, also $w_1 = 0$ and $w_2 = \infty$ are symmetrical to the image circles $T(G)$ and $T(K)$. Hence $T(G)$ and $T(K)$ must be circles around the origin, because z_2 does not lie in G or K and therefore also $T(z_2) = \infty$ does not lie on $T(G)$ or $T(K)$.

Being $z_3 = -1$ in G, from $T(-1) = -1$ G is mapped onto the unit circle.

From $T(-1) = -1 = k \cdot \dfrac{-1 - z_1}{-1 - z_2} = -k$ it follows $k = 1$ and

$$T(z) = \frac{z - z_1}{z - z_2}.$$

For $z_1 = 1$ and $z_2 = -3$ results $\quad T_1(z) = \dfrac{z - 1}{z + 3}$.

For $z_1 = -3$ and $z_2 = 1$ results $\quad T_2(z) = \dfrac{z + 3}{z - 1}$.

As a check, we verify that $T_1(G)$ is the unit circle:

$$|T_1(-1 + it)| = \left|\frac{-1 + it - 1}{-1 + it + 3}\right| = \frac{\sqrt{4 + t^2}}{\sqrt{4 + t^2}} = 1\,.$$

Since $z_4 = 2 + \sqrt{5}$ lies on K, the radius R of $T(K)$ can be computed via $R = |T(2 + \sqrt{5})|$.

Specifically, for T_1 it results

$$R_1 = \left|\frac{2+\sqrt{5}-1}{2+\sqrt{5}+3}\right| = \left|\frac{1+\sqrt{5}}{5+\sqrt{5}}\right| = \frac{1}{\sqrt{5}} \approx 0.447$$

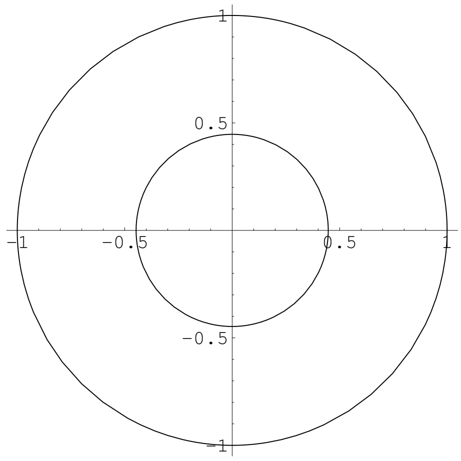

Figure 1 c): Circumferences $T_1(G)$ and $T_1(K)$

Since z_1 lies in the interior of K, the domain on the right to G and external to the circular disc K is mapped onto the circular ring $\{w \in \mathbb{C} \mid R_1 < |w| < 1\}$.

In the other case, i.e. with T_2, one gets $R_2 = 1/R_1 = \sqrt{5} \approx 2.236$ and the circular ring $\{w \in \mathbb{C} \mid 1 < |w| < \sqrt{5}\}$. From this, the interior of K is mapped onto the external region and the plane lying to the left of G is mapped into the interior of the unit circle.

Exercise 2:

Consider the half-plane E lying to the right of the line $G = \{z \in \mathbb{C} | z = -1 + it, t \in \mathbb{R}\}$ and outside the circular disc $K = \{z \in \mathbb{C} \mid |z-2| \leq \sqrt{5}\}$.

Compute a function harmonic in E and such that it has value 1 on the boundary of K and 0 on G.

Hint: Transform the problem as given in Exercise 1, solve the conformal transformed problem in polar coordinates and then transform back.

Solution:

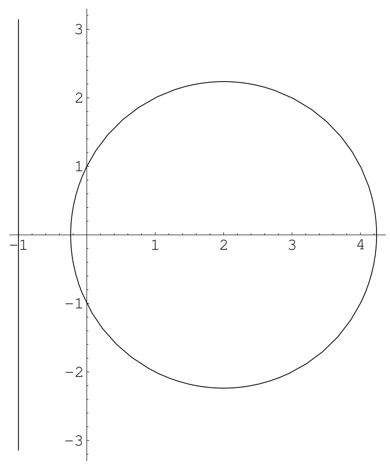

Figure 2 a): Half-plane E without the circular disc K

We look for the real-valued function $u : E \subset \mathbb{R}^2 \to \mathbb{R}$ for which with $z = x + iy$ it holds

$$\begin{aligned} \Delta u(x,y) &= 0 \quad \text{for all} \quad (x,y)^T \in E \\ u(x,y) &= 0 \quad \text{for all} \quad (x,y)^T \in G \\ u(x,y) &= 1 \quad \text{for all} \quad (x,y)^T \in K . \end{aligned}$$

The domain E is now mapped, for example through the conformal Möbius transformation

$$T_2(z) = \frac{z+3}{z-1}$$

from Exercise 1, into the circular ring

$$T_2(E) = \{w \in \mathbb{C} \mid 1 < |w| < \sqrt{5}\}.$$

The transformed conformal function with $T_2(z) = w = \xi + i\eta$ reads

$$U(\xi, \eta) = U(w) := u(T_2^{-1}(w))$$

and in light of the second theorem of conformal transformation $\Delta u = \Delta U \cdot |T_2'(z)|^2$ and $T_2'(z) \neq 0$ it satisfies the boundary value problem

$$\begin{aligned} \Delta U(\xi,\eta) &= 0 \quad \text{for all} \quad (\xi,\eta)^T \in T_2(E) \\ U(\xi,\eta) &= 0 \quad \text{for all} \quad (\xi,\eta)^T \in T_2(G) \\ U(\xi,\eta) &= 1 \quad \text{for all} \quad (\xi,\eta)^T \in T_2(K)\,. \end{aligned}$$

Such problem in the circular ring $T_2(E)$ is better solvable in polar coordinates, thus with $\xi = r\cos\varphi\,,\ \eta = r\sin\varphi\,,\ 1 \le r \le \sqrt{5}\,,$
$0 \le \varphi < 2\pi$ and $v(r,\varphi) := U(\xi(r,\varphi),\eta(r,\varphi))$ is transformed and the problem reads now

$$\begin{aligned} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi} &= 0 \quad \text{for all} \quad 1 \le r \le \sqrt{5}\,, \quad 0 \le \varphi < 2\pi \\ v(\sqrt{5},\varphi) &= 1\,, \quad 0 \le \varphi < 2\pi \\ v(1,\varphi) &= 0\,. \end{aligned}$$

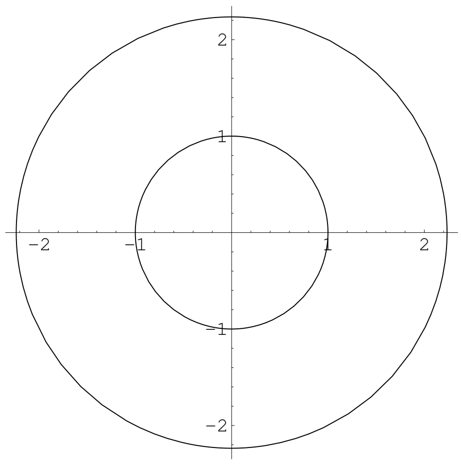

Figure 2 b): $T_2(E\backslash K) = \{w \in \mathbb{C} \mid 1 < |w| < \sqrt{5}\}$

Because of the symmetry of the domain and the rotational symmetry of the boundary data, it is reasonable to assume that such problem possesses a rotation-symmetric solution, meaning $v(r,\varphi) = v(r)$. Thus the problem simplifies into a boundary problem with standard differential equation

$$\begin{aligned} v_{rr} + \frac{1}{r}v_r &= 0 \quad \text{for all} \quad 1 < r < \sqrt{5} \\ v(\sqrt{5}) &= 1\,, \\ v(1) &= 0\,, \end{aligned}$$

whose general solution is $v(r) = c_1 \ln r + c_2$. Inserting the boundary values it returns $0 = v(1) = c_1 \ln 1 + c_2 = c_2$ and $1 = v(\sqrt{5}) = c_1 \ln\sqrt{5} \Rightarrow c_1 = 1/\ln\sqrt{5}\,.$

Thus the required solution in polar coordinates reads

$$v(r,\varphi) = v(r) = \frac{\ln r}{\ln\sqrt{5}}\,.$$

The inverse transformation in the $w-$ plane gives the solution

$$U(w) = \frac{\ln|w|}{\ln\sqrt{5}}\,.$$

The inverse transformation in the $z-$ plane gives the solution

$$u(z) = \frac{\ln|T_2(z)|}{\ln\sqrt{5}}\,.$$

With $z = x + iy$ and

$$|T_2(z)| \;=\; \left|\frac{z+3}{z-1}\right| = \left(\frac{(x+3)^2+y^2}{(x-1)^2+y^2}\right)^{1/2} \;=\; \left(\frac{x^2+y^2+6x+9}{x^2+y^2-2x+1}\right)^{1/2}$$

it returns the solution representation in the $(x,y)-$ plane

$$u(x,y) = \frac{1}{2\ln\sqrt{5}}\ln\left(\frac{x^2+y^2+6x+9}{x^2+y^2-2x+1}\right)\,.$$

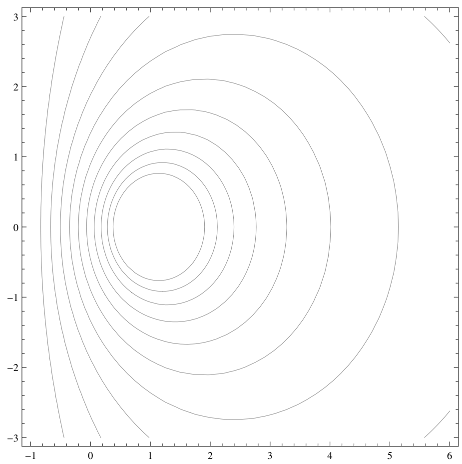

Figure 2 c): Contour lines of the solution $u(x,y)$

Hand in until: 9.6.