Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Exercise class 4

Exercise 1:

Let the mapping $T: \mathbb{C}^* \to \mathbb{C}^*$ with

$$T(z) = \frac{z+2}{z-2}$$

be given.

a) Does T represent a Möbius transformation?

b) Compute the inverse mapping.

c) Determine the image of the real axis.

d) Determine the image of the circumference $|z| = 2$.

e) Determine the image of the imaginary axis.

f) Where is the semicircle H mapped to?

$$H := \{z \in \mathbb{C} \mid |z| \leq 2\,,\ \mathrm{Im}(z) \geq 0\}$$

Solution:

$$T(z) = \frac{z+2}{z-2} \qquad \left(= \frac{az+b}{cz+d}\right).$$

a) T is a Möbius transformation, since $ad - bc = 1 \cdot (-2) - 2 \cdot 1 = -4 \neq 0$.

b) By solving $w = \dfrac{z+2}{z-2}$ with respect to z one gets the inverse mapping:

$$w(z-2) = z+2 \Rightarrow -2w - 2 = z(1-w) \Rightarrow z = T^{-1}(w) = \frac{2w+2}{w-1}$$

c) For the real axis it holds $z = x \in \mathbb{R}$. From this one obtains

$$T(z) = T(x) = \frac{x+2}{x-2} \in \mathbb{R}\,.$$

Thus the real axis is mapped onto itself.

d) Being $T(2) = \infty$ (the image is a line), $T(-2) = 0$ (through zero) and $T(2i) = -i$, its image is the imaginary axis.

e) Since the imaginary axis is symmetrical to the real axis and to the circumference $|z| = 2$, such symmetry is preserved in the image space.

Therefore the image of the imaginary axis is a circumference around zero, with radius $R = |T(2i)| = |-i| = 1$, which is the unit circle.

f) The circular disc $|z| \leq 2$ is mapped by $T(0) = -1$ onto the left half-plane, i.e. onto $\mathrm{Re}(w) \leq 0$.

The upper half-plane is mapped by $T(2i) = -i$ onto the lower half-plane, i.e. onto $\mathrm{Im}(w) \leq 0$.

Thus,

$$H := \{z \in \mathbb{C} \mid |z| \leq 2\,,\ \mathrm{Im}(z) \geq 0\}$$

is mapped onto the third quadrant:

$$Q_3 := \{w \in \mathbb{C} \mid \mathrm{Re}(w) \leq 0\,,\ \mathrm{Im}(w) \leq 0\}\ .$$

Exercise 2:

We are looking for a Möbius transformation $w = T(z)$ with $T(-1) = 1$ and $T(0) = 0$, which maps the left half-space $\mathrm{Re}\,(z) \le 0$ onto the circular disc $|w-1| \le R$. How large is R?

Solution:

The idea of solution is that the imaginary axis is mapped onto the circumference $|w-1| = R$.

$z_1 = -1$ is mapped onto the centre $w_1 = 1$ of the circular disc $|w-1| \le R$. The point $z_3 = 1$, symmetrical point of z_1 with respect to the imaginary axis, is mapped onto the point of the image circle $w_3 = \infty$ that is symmetrical to $w_1 = 1$.

Thus the imaginary axis is mapped onto $|w-1| = R$.

Then the Möbius transformation $w = T(z)$ results from the three-points formula

$$\frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2} = \frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2}$$

with $z_2 = 0$ and $T(z_2) = w_2 = 0$. One gets

$$\frac{z-(-1)}{z} : \frac{1-(-1)}{1} = \frac{w-1}{w} : \frac{w_3-1}{w_3}\bigg|_{w_3\to\infty} \Rightarrow \frac{z+1}{2z} = \frac{w-1}{w}$$

$$\Rightarrow w(z+1) = (w-1)2z \Rightarrow w(-z+1) = -2z \Rightarrow w = T(z) = \frac{2z}{z-1}$$

$z_2 = 0$ lies on the imaginary axis and is mapped onto the image circumference $|w-1| = R$. From this one gets the radius

$$R = |T(0) - 1| = 1\,.$$

$z_1 = -1$ lies on the left half-plane and is mapped onto the centre of the circular disc $|w-1| \le 1$. Then by continuity the left half-plane is mapped onto the circular disc.

Dates of classes: 22.5. - 26.5.