Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Homework 4

Exercise 1:

For the inverse $w = f(z) := \frac{1}{z}$ with $z \neq 0$ determine the image

a) of the line $\text{Re}(z) = 2$,

b) of the ray $\text{Re}(z) > 0 \wedge \text{Im}(z) = 0$,

c) of the circumference $|z| = 3$,

d) of the circumference $|z - 2i| = 2$ and

e) of the circumference $|z - 2i| = 1$.

Solution:

The reverse mapping of the inverse $w = f(z) = \frac{1}{z}$ is

$$z = f^{-1}(w) = \frac{1}{w} ,$$

where $z \neq 0$ and $w \neq 0$. This returns the following images

a) $2 = \text{Re}(z) = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{\bar{w}}\right) \quad \Leftrightarrow \quad 4w\bar{w} - w - \bar{w} = 0$

$$\Leftrightarrow \frac{1}{16} = w\bar{w} - \frac{1}{4}w - \frac{1}{4}\bar{w} + \frac{1}{16} \Leftrightarrow \left(w - \frac{1}{4}\right)\left(\bar{w} - \frac{1}{4}\right) = \frac{1}{16}$$

$$\Leftrightarrow \left|w - \frac{1}{4}\right| = \frac{1}{4}$$

Image of $\text{Re}(z) = 2$ is the circumference around $w_0 = \frac{1}{4}$ with radius $r = \frac{1}{4}$.

b) $\mathrm{Re}(z) > 0 \wedge \mathrm{Im}(z) = 0 \quad \Leftrightarrow \quad z = x > 0 \quad \Leftrightarrow \quad w = \frac{1}{z} = \frac{1}{x}$

The ray is mapped onto itself, only running in the opposite direction.

c) $3 = |z| = \left|\frac{1}{w}\right| \quad \Leftrightarrow \quad |w| = \frac{1}{3}\,.$

The circumference centered in zero and radius 3 is mapped onto the circumference centered in zero and radius $\frac{1}{3}$.

d) $|z - 2i| = 2 \quad \Leftrightarrow \quad 4 = z\bar{z} + 2iz - 2i\bar{z} + 4 = \frac{1}{w}\frac{1}{\bar{w}} + 2i\frac{1}{w} - 2i\frac{1}{\bar{w}} + 4$

$\Leftrightarrow \quad -2i(w - \bar{w}) = -1 \quad \Leftrightarrow \quad \mathrm{Im}(w) = -\frac{1}{4}\,.$

The image of the circumference is the line $\mathrm{Im}(w) = -\frac{1}{4}$.

e) $|z - 2i| = 1 \quad \Leftrightarrow \quad 1 = z\bar{z} + 2iz - 2i\bar{z} + 4 = \frac{1}{w}\frac{1}{\bar{w}} + 2i\frac{1}{w} - 2i\frac{1}{\bar{w}} + 4$

$\Leftrightarrow\ w\bar{w} + \frac{2i}{3}\bar{w} - \frac{2i}{3}w + \frac{4}{9} = \frac{4}{9} - \frac{1}{3} \ \Leftrightarrow\ \left|w + \frac{2i}{3}\right| = \frac{1}{3}\,.$

The image of the circumference is the circumference around $w_0 = -\frac{2i}{3}$ with radius $r = \frac{1}{3}$.

Exercise 2:

Let the points

$$z_1 = 1,\ z_2 = 1 + 2i,\ z_3 = i$$

and

$$w_1 = 0,\ w_2 = 1 + i,\ w_3 = -1 - i$$

be given.

a) Compute the Möbius transformation T such that for $j = 1, 2, 3$ it holds:

$$w_j = T(z_j)\,.$$

b) Do $z_0 = 2 + i$ and z_1, z_2, z_3 lie on a (generalized) circle K?

c) Do $w_0 = T(z_0)$ and w_1, w_2, w_3 lie on a (generalized) circle $T(K)$?

Solution:

a) Since z_j , $j = 1, 2, 3$ are different from each other and this also holds for w_j , $j = 1, 2, 3$, there exists exactly a Möbius transformation T with $w_j = T(z_j)$, which can be obtained by solving the three-points formula

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}\,.$$

with respect to w. The substitution returns:

$$\frac{w}{w - 1 - i} : \underbrace{\frac{-1 - i}{-1 - i - 1 - i}}_{=1/2} = \frac{z - 1}{z - 1 - 2i} : \underbrace{\frac{i - 1}{i - 1 - 2i}}_{=1/i} \quad \Rightarrow$$

$$\frac{2w}{w - 1 - i} = \frac{i(z - 1)}{z - 1 - 2i} \quad \Rightarrow \quad 2w(z - 1 - 2i) = i(z - 1)(w - 1 - i) \quad \Rightarrow$$

$$w(2z - 2 - 4i - i(z - 1)) = (1 - i)(z - 1) \quad \Rightarrow$$

$$w = \frac{(1 - i)z - (1 - i)}{(2 - i)z - 2 - 3i} =: T(z)$$

b) The cross-ratio for $z_0 = 2 + i$ returns

$$\frac{z_0 - z_1}{z_0 - z_2} : \frac{z_3 - z_1}{z_3 - z_2}$$

$$= \frac{2 + i - 1}{2 + i - 1 - 2i} : \frac{i - 1}{i - 1 - 2i} = \frac{(1 + i)(-1 - i)}{(1 - i)(i - 1)} = -1 \in \mathbb{R}$$

Therefore $z_0, \ldots, z_3$ lie on the same circle.

With a sampling one finds out that it is exactly the circumference $|z - 1 - i| = 1$.

c) Answer 1:

Since Möbius transformations map generalized circles onto generalized circles, also $w_0, \ldots, w_3$ lie on the same generalized circle (here: the angle bisector).

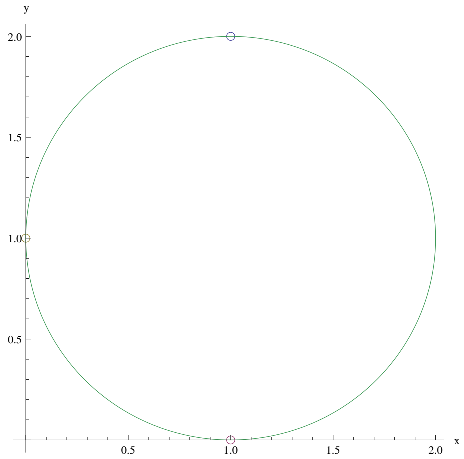

Bild 1: z_1, z_2, z_3 und K

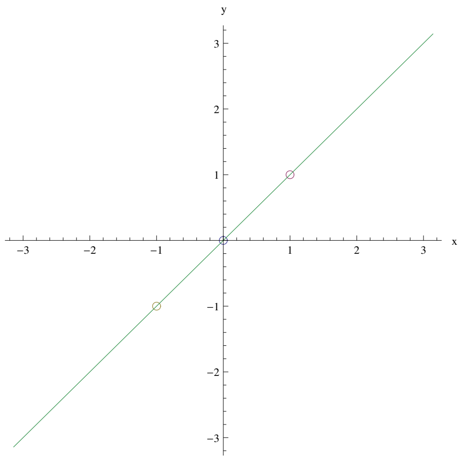

Bild 2: w_1, w_2, w_3 und $T(K)$

Antwort 2:

Since T results from the three-points formula and it is $w_0 = T(z_0)$, from the cross-ratio of $w_0, \ldots, w_3$ one gets

$$\frac{w_0 - w_1}{w_0 - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z_0 - z_1}{z_0 - z_2} : \frac{z_3 - z_1}{z_3 - z_2} = -1 \in \mathbb{R}\,,$$

meaning that the cross-ratio is preserved under T.

Hand in until: 26.5.