Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Exercise class 3

Exercise 1:
Determine the image of

$$K := \left\{ z \in \mathbb{C} \mid 0 \leq \text{Re}(z) \, , \, 0 \leq \text{Im}(z) \, , \, \text{Re}(z)^2 + \text{Im}(z)^2 \leq 1 \right\}$$

under the mapping defined by $f(z) = ((1+i)z)^2$.

Solution:

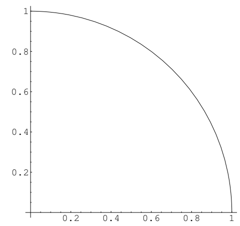

Image 1.1 $K := \{z \in \mathbb{C} \mid 0 \leq \text{Re}(z) \, , \, 0 \leq \text{Im}(z) \, , \, \text{Re}(z)^2 + \text{Im}(z)^2 \leq 1\}$

The mapping $f(z) = ((1+i)z)^2$ is interpreted as composition $f = f_2 \circ f_1$, with $f_1(z) = (1+i)z$ and $f_2(u) = u^2$.

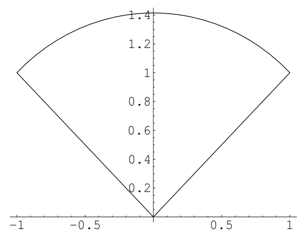

Image 1.2 $f_1(K)$

The function $f_1(z) = (1+i)z = \sqrt{2}e^{\pi i/4} r e^{\varphi i} = r\sqrt{2}e^{(\pi/4+\varphi)i}$ determines a scaling of factor $\sqrt{2}$ and a rotation of angle $\varphi = \dfrac{\pi}{4}$.

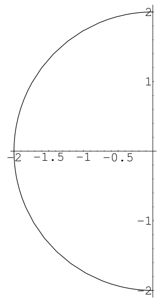

Image 1.3 $f(K) = f_2(f_1(K))$

The function $f_2(u) = u^2 = \left(re^{\psi i}\right)^2 = r^2 e^{2\psi i}$ doubles the angle and squares the distance from the origin.

Alternative composition of f :

For $f(z) = ((1+i)z)^2 = (1+i)^2 z^2 = 2iz^2$ we can alternatively read $f = f_4 \circ f_3$ with $f_3(z) = z^2$ and $f_4(u) = 2iu$. Then it results

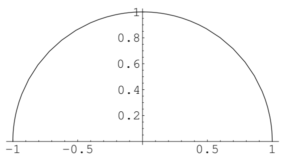

Image 1.4 $f_3(K)$

The function $f_4(u) = 2iu = e^{\pi i/2} r e^{\varphi i} = 2re^{(\pi/2+\varphi)i}$ determines a scaling of factor 2 and a rotation of angle $\varphi = \dfrac{\pi}{2}$.

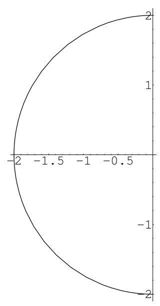

Image 1.5 $f(K) = f_4(f_3(K))$

Exercise 2:

For the exponential function exp determine the images of the following sets

a) $D_1 = \left\{z \in \mathbb{C} \mid 0 \le \mathrm{Re}(z) \le \ln(10)\,,\; -\frac{\pi}{2} \le \mathrm{Im}(z) \le \frac{\pi}{2}\right\}$,

b) $D_2 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \le 0\,,\; 0 \le \mathrm{Im}(z) < \pi\}$,

c) $D_3 = \{z \in \mathbb{C} \mid -\pi < \mathrm{Im}(z) < \pi\}$.

Solution:

a)

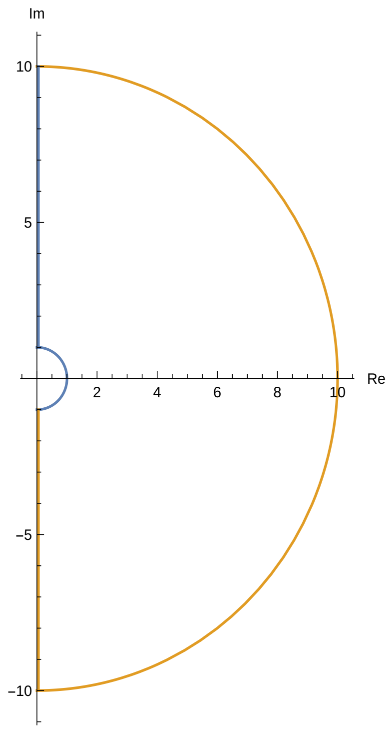

Image 2 a) Semicircular ring $\exp(D_1)$

b)

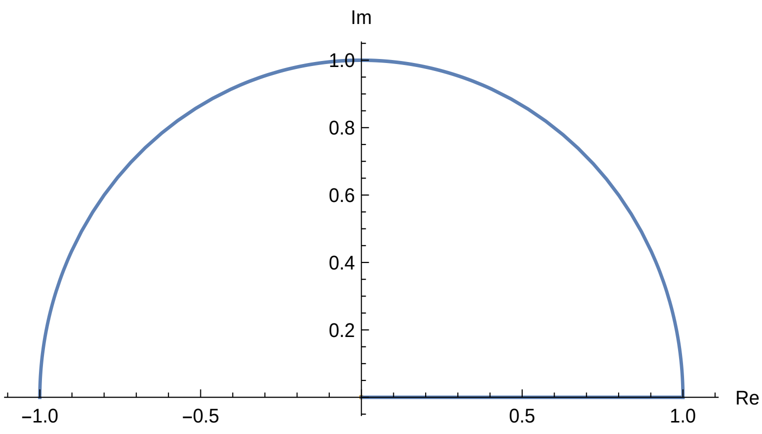

Image 2 b) Semicircle $\exp(D_2)$

c) Sliced complex plane: $\exp(D_3) = \mathbb{C}^-$

Exercise 3:

a) Given $z_1 = 2 + \dfrac{\pi i}{3}$ and $z_2 = -1 + \dfrac{2\pi i}{3}$, compute

$$\exp(z_1)\,,\ \exp(z_2) \quad \text{and} \quad \exp(z_1 + z_2)$$

in Carthesian coordinates and verify with such example the validity of the functional equation for the exp function in $\mathbb{C}$:

$$\exp(z_1) \cdot \exp(z_2) = \exp(z_1 + z_2)\,.$$

b) For the principal value of the complex logarithm log, with $z_1 = -1 - i\sqrt{3}$ and $z_2 = -2i$ one compute

$$\log(z_1)\,,\ \log(z_2) \quad \text{and} \quad \log(z_1 z_2),$$

and check with such example whether for the principal part it holds:

$$\log(z_1) + \log(z_2) = \log(z_1 z_2)\,.$$

Solution:

a) $\exp(z_1) = \exp\left(2 + \dfrac{\pi i}{3}\right) = e^2\left(\cos\left(\dfrac{\pi}{3}\right) + i\sin\left(\dfrac{\pi}{3}\right)\right) = \dfrac{e^2}{2} + i\dfrac{e^2\sqrt{3}}{2}$

$\exp(z_2) = \exp\left(-1 + \dfrac{2\pi i}{3}\right) = e^{-1}\left(\cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)\right) = -\dfrac{1}{2e} + i\dfrac{\sqrt{3}}{2e}$

$\exp(z_1 + z_2) = \exp(1 + \pi i) = e\left(\cos(\pi) + i\sin(\pi)\right) = -e$

From this one gets

$$\exp(z_1)\exp(z_2) = e^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\frac{1}{e}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -e = \exp(z_1 + z_2)\,.$$

b) The principal value $\log(z)$ of the logarithm (shortened $\ln(z)$) is defined for $-\pi < \arg(z) < \pi$ by

$$\log(z) = \ln|z| + i\arg(z)\,.$$

$\log(z_1) = \log(-1 - i\sqrt{3}) = \ln|-1 - i\sqrt{3}| + i\arg(-1 - i\sqrt{3}) = \ln(2) - i\dfrac{2\pi}{3}$

$\log(z_2) = \log(-2i) = \ln|-2i| + i\arg(-2i) = \ln(2) - i\dfrac{\pi}{2}$

$\log(z_1) + \log(z_2) = \ln(2) - i\dfrac{2\pi}{3} + \ln(2) - i\dfrac{\pi}{2} = \ln(4) - i\dfrac{7\pi}{6}$

$z_1 z_2 = -2i(-1 - i\sqrt{3}) = -2\sqrt{3} + 2i$

$\log(z_1 z_2) = \log(-2\sqrt{3}+2i) = \ln|-2\sqrt{3}+2i| + i\arg(-2\sqrt{3}+2i)$
$= \ln(4) + i\left(\pi - \frac{\pi}{6}\right) = \ln(4) + i\frac{5\pi}{6}$
The functional equation of the logarithm function

$$\log(z_1) + \log(z_2) = \ln(4) - i\frac{7\pi}{6} \neq \ln(4) + i\frac{5\pi}{6} = \log(z_1 z_2)$$

is not satisfied for the principal value in such example.

Though the angles $-\frac{7\pi}{6}$ and $\frac{5\pi}{6}$ in principle describe the same angle in the circle, they differ by a whole rotation of 2π. This implies that $\log(z_1)+\log(z_2)$ leads to a sub-branch of the complex logarithm. For the inverse function $\exp$ of course it applies

$$\exp\left(\ln(4) - i\frac{7\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = z_1 z_2 = \exp\left(\ln(4) + i\frac{5\pi}{6}\right) .$$

Dates of classes: 2.5.-5.5.