Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Homework 3

Exercise 1:

The cos function in the complex field is defined by

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \ .$$

Compute real and imaginary part of $\cos z$ and determine all solutions of $\cos z = 3$.

Solution:

With $z = x + iy$ it is:

$$\begin{aligned} \cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) \\ &= \frac{1}{2}\left(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right) \\ &= \cos x \cdot \frac{1}{2}(e^y + e^{-y}) - i\sin x \frac{1}{2}(e^y - e^{-y}) \\ &= \cos x \cosh y - i \sin x \sinh y \end{aligned}$$

$$3 = \cos z = \cos x \cosh y - i \sin x \sinh y \quad \Rightarrow \quad \sin x \sinh y = 0$$

Case 1:

$$\sinh y = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad 3 = \cos x \cosh 0 = \cos x \qquad \text{no solution}$$

Case 2 :

$$\sin x = 0 \ \Rightarrow \quad x = k\pi \ \Rightarrow \quad 3 = \cos(k\pi)\cosh y \ \Rightarrow \quad k = 2n \ , \ y = \pm \text{arcosh } 3$$

$$\Rightarrow \quad z_n = 2n\pi \pm i \, \text{arcosh } 3 \ , \quad n \in \mathbb{Z}$$

Exercise 2:

Let the Joukowski function $w = f(z) := \dfrac{1}{2}\left(\dfrac{z}{4} + \dfrac{4}{z}\right)$ be given.

a) Determine the images

  (i) of the circumference $|z| = 5$,

  (ii) of the ray $\mathrm{Re}(z) < 0$, $\mathrm{Im}(z) = 0$,

  (iii) of the ray $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) < 0$.

b) Compute the inverse function $z = f^{-1}(w)$ for $|z| > 4$.

Solution:

a) (i) With the polar representation $z = 5e^{i\varphi}, 0 \le \varphi < 2\pi$ of the circumference $|z| = 5$ on gets:

$$\begin{aligned} f(z) &= \frac{1}{2}\left(\frac{5e^{i\varphi}}{4} + \frac{4}{5e^{i\varphi}}\right) \\ &= \frac{1}{2}\left(\frac{5}{4}(\cos\varphi + i\sin\varphi) + \frac{4}{5}(\cos\varphi - i\sin\varphi)\right) \\ &= \underbrace{\left(\frac{5}{8} + \frac{4}{10}\right)\cos\varphi}_{=u} + i\underbrace{\left(\frac{5}{8} - \frac{4}{10}\right)\sin\varphi}_{=v}. \end{aligned}$$

$u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ satisfy the ellipse equation

$$\frac{u^2}{\left(\frac{5}{8} + \frac{4}{10}\right)^2} + \frac{v^2}{\left(\frac{5}{8} - \frac{4}{10}\right)^2} = \cos^2\varphi + \sin^2\varphi = 1 .$$

The circumference $|z| = 5$ is therefore mapped onto an ellipse.

Mathematica plot code

```
ParametricPlot[{(5/8 + 4/10) Cos[t],
(5/8 - 4/10) Sin[t]}, {t, 0, 2 Pi},
AxesLabel -> {"Re", "Im"}, PlotRange -> {-1, 1}]
```

Semi-axes:

$$a = \frac{5}{8} + \frac{4}{10} = \frac{41}{40} = 1.025\,, \qquad b = \frac{5}{8} - \frac{4}{10} = \frac{9}{40} = 0.225$$

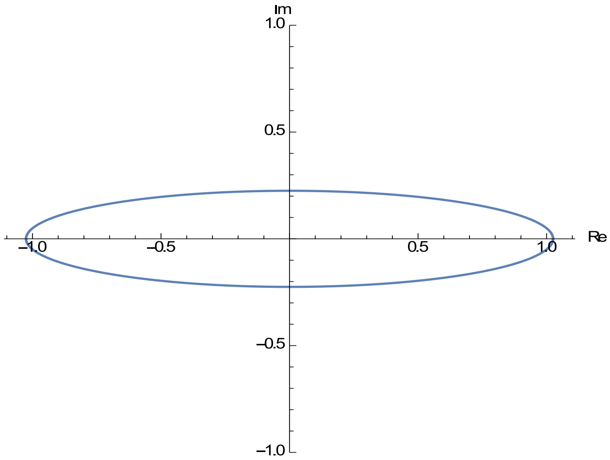

Image 2 a)(i): Ellipse

(ii) From the polar representation $z = re^{i\pi} = -r,\ 0 < r < \infty$ of the ray $\mathrm{Re}(z) < 0$, $\mathrm{Im}(z) = 0$ it follows:

$$f(z) = \frac{1}{2}\left(\frac{-r}{4} + \frac{4}{-r}\right) = -\frac{1}{2}\left(\frac{r}{4} + \frac{4}{r}\right) =: g(r)$$

It is $\lim_{r\to 0+} g(r) = -\infty = \lim_{r\to\infty} g(r)$ and

$$0 \geq -\left(\frac{r}{2} - 2\right)^2 = -\frac{r^2}{4} + 2r - 4 \quad \Leftrightarrow \quad -\frac{1}{2}\left(\frac{r}{4} + \frac{4}{r}\right) \leq -1\,.$$

$r = 4$ attains the maximum of g with $g(4) = -1$. Of course it also holds

$$g'(r) = -\frac{1}{2}\left(\frac{1}{4} - \frac{4}{r^2}\right) = -\frac{r^2 - 16}{8r^2} = 0 \;\Rightarrow\; r_{1,2} = \pm 4 \;\Rightarrow\; r = 4\,.$$

The image interval $]-\infty, -1]$ is therefore passed through twice.

(iii) From the polar representation $z = re^{i3\pi/2} = -ir,\ 0 < r < \infty$ of the ray $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) < 0$ it follows:

$$f(z) = \frac{1}{2}\left(\frac{-ir}{4} + \frac{4}{-ir}\right) = -i\underbrace{\left(\frac{r}{8} - \frac{2}{r}\right)}_{=t} \quad \text{with} \quad t \in \mathbb{R}$$

The image is thus the imaginary axis.

b) The inverse function of f is obtained by solving $w = f(z)$ with respect to z:

$$w = \frac{1}{2}\left(\frac{z}{4} + \frac{4}{z}\right) \;\Rightarrow\; 8wz = z^2 + 16$$

$$\Rightarrow\; z^2 - 8wz + 16 = (z - 4w)^2 - 16w^2 + 16 = 0 \;\Rightarrow\; (z - 4w)^2 = 16(w^2 - 1)$$

$$\Rightarrow\; z = f^{-1}(w) = 4(w + \sqrt{w^2 - 1})$$

For $\sqrt{w^2 - 1}$ it is necessary to select the branch for which $|z| > 4$ applies.

Hand in until: 5.5.