**Fachbereich Mathematik der Universität Hamburg** **SoSe 2023**
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# Complex functions for Engineering Students

## Solutions of Exercise class 2

**Exercise 1:**

Draw the following sets of points in the complex plane:

a) $\{z \in \mathbb{C} : |3z + 6 - i| = 9\}$,

b) $\{z \in \mathbb{C} : \text{Re}(z) \le \text{Im}(z)\}$,

c) $\{z \in \mathbb{C} : \text{Re}((1-i)z) = 2\}$,

d) $\{z \in \mathbb{C} : \pi \le \arg(z) \le 3\pi/2\,,\ 4 \le |z| \le 5\}$.

**Solution:**

a) $|3z + 6 - i| = 9 \quad \Leftrightarrow \quad \left| z + 2 - \frac{i}{3} \right| = 3$

With the Cartesian representation $z = x + iy$ one obtains:

$$\begin{aligned} \left| z + 2 - \frac{i}{3} \right| &= \left| x + iy + 2 - \frac{i}{3} \right| = \left| x + 2 + i\left(y - \frac{1}{3}\right) \right| \\ &= \sqrt{(x+2)^2 + \left(y - \frac{1}{3}\right)^2} = 3 \end{aligned}$$

$$\Leftrightarrow \quad (x+2)^2 + \left(y - \frac{1}{3}\right)^2 = 3^2$$

Circle of radius $r = 3$ around the point $\left(-2, \frac{1}{3}\right)$.

b) $\{z \in \mathbb{C} : \text{Re}(z) \le \text{Im}(z)\}$

Witht $z = x + iy$ one gets $x \le y$, that is the half-plane above the bisecting line $x = y$.

c)
$$\begin{aligned} 2 &= \text{Re}((1-i)z) = \text{Re}((1-i)(x+iy)) \\ &= \text{Re}(x + y + i(-x + y)) = x + y \end{aligned}$$

The set of points is described by the following line:

$$y = -x + 2\,.$$

d) $\{z \in \mathbb{C} : \pi \leq \arg(z) \leq 3\pi/2\,,\ 4 \leq |z| \leq 5\}$ is the quarter of circle in the third quadrant with radius $4 \leq r \leq 5$.

**Exercise 2:**

a) For $z \in \mathbb{C}$ consider the polynomial $p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with real coefficients $a_0, \ldots a_n$.

Show that if $z_0 \in \mathbb{C}$ is a root of $p$, then also $\bar{z}_0$ is a root of $p$.

b) Prove that the circle $|z - z_0| = r$ in the complex plane has the following representation

$$z\bar{z} - z\bar{z}_0 - z_0\bar{z} + z_0\bar{z}_0 = r^2 \quad \text{with} \quad z, z_0 \in \mathbb{C}.$$

c) Determine the curve described by

$$z\overline{z} = (4 - 3i)\overline{z} + (4 + 3i)z + 144.$$

**Solution:**

a)
$$\begin{aligned} p(\bar{z}_0) &= a_n(\bar{z}_0)^n + a_{n-1}(\bar{z}_0)^{n-1} + \cdots + a_1\bar{z}_0 + a_0 \\ &= a_n\overline{z_0^n} + a_{n-1}\overline{z_0^{n-1}} + \cdots + a_1\overline{z_0} + a_0 \\ &= \overline{a_n z_0^n} + \overline{a_{n-1} z_0^{n-1}} + \cdots + \overline{a_1 z_0} + \bar{a}_0 \;=\; \overline{p(z_0)} = \overline{0} = 0 \end{aligned}$$

b) $|z - z_0| = r \overset{\geq 0}{\Leftrightarrow} r^2 = |z - z_0|^2 = (z - z_0)(\bar{z} - \bar{z}_0) = z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + \bar{z}_0 z_0$

c) Comparing it to the circle representation we obtain

$z_0 = (4 - 3i) \quad$ and $\quad r^2 - z_0\bar{z}_0 = 144$

$\Rightarrow \quad z_0\bar{z}_0 = 25 \quad \Rightarrow \quad r^2 = z_0\bar{z}_0 + 144 = 25 + 144 = 169$

Thus the curve describes a circle with center $z_0 = (4 - 3i)$ and with radius $r = 13$.

**Exercise 3:**

Analyze the convergence of the sequence

$$z_0 = 3\,, \qquad z_{n+1} = \frac{3-2i}{4}\,(1+2i+z_n)$$

and if possible determine its limit value.

**Solution:**

In case $z_n$ converges, then with $z^* := \lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} z_{n+1}$ it is:

$$z^* = \frac{3-2i}{4}\,(1+2i+z^*)$$

$$\Rightarrow \quad z^*\left(1-\frac{3-2i}{4}\right) = z^*\left(\frac{1+2i}{4}\right) = \frac{(1+2i)(3-2i)}{4}$$

$$\Rightarrow \quad z^* = 3-2i.$$

$z_n$ converges since

$$\begin{aligned}
|z_{n+1}-z^*| &= |z_{n+1}-(3-2i)| = \left|\frac{3-2i}{4}\,(1+2i+z_n)-(3-2i)\right| \\
&= \left|\frac{3-2i}{4}\right| |1+2i+z_n-4| = \frac{\sqrt{13}}{4}\,|z_n-(3-2i)| \\
&= \left(\frac{\sqrt{13}}{4}\right)^2 |z_{n-1}-(3-2i)| \\
&\;\;\vdots \\
&= \left(\frac{\sqrt{13}}{4}\right)^{n+1} |z_0-(3-2i)| = 2\left(\frac{\sqrt{13}}{4}\right)^{n+1} \xrightarrow{n\to\infty} 0
\end{aligned}$$

**Dates of classes:** 17.4. - 21.4.