Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Complex functions for Engineering Students

Solutions of Homework 2

Exercise 1:

Let the complex numbers $z_1 := \dfrac{5 - i\sqrt{3}}{1 - i\sqrt{3}} - 1$ and $z_2 := -1 + i$ be given.

a) Determine real and imaginary part of z_1 and the polar representation of z_1 and z_2.

b) Compute z_2^{12}.

c) Provide all solutions of the equation $(w - z_2)^4 = -64$ in Cartesian coordinates.

Solution:

a) $z_1 = \dfrac{5 - i\sqrt{3}}{1 - i\sqrt{3}} - 1 = \dfrac{4}{1 - i\sqrt{3}} = 1 + i\sqrt{3}$

$\Rightarrow \quad \text{Re}\ (z_1) = 1\,, \quad \text{Im}\ (z_1) = \sqrt{3}$

$|z_1| = \sqrt{1+3} = 2,$

$\arg\ z_1 = \arctan\dfrac{\sqrt{3}}{1} = \dfrac{\pi}{3}, \qquad z_1 = 2e^{\pi i/3},$

$z_2 = -1 + i \quad \Rightarrow \quad \text{Re}\ (z_2) = -1\,, \quad \text{Im}\ (z_2) = 1$

$|z_2| = \sqrt{2}, \quad \arg\ z_2 = \pi + \arctan\dfrac{1}{-1} = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

$z_2 = \sqrt{2}e^{3\pi i/4}$

b) $z_2^{12} = \left(\sqrt{2}e^{3\pi i/4}\right)^{12} = (\sqrt{2})^{12}\left(e^{3\pi i/4}\right)^{12} = 64e^{9\pi i} = -64$

c) $(w - z_2)^4 = -64 = 64e^{\pi i}$

$\Rightarrow \quad w_k = z_2 + 2\sqrt{2}e^{(\pi i + 2k\pi i)/4} = z_2 + 2\sqrt{2}e^{(2k+1)\pi i/4}\,, \quad k = 0, 1, 2, 3$

$w_0 = z_2 + 2\sqrt{2}e^{\pi i/4} = -1 + i + 2(1+i) = 1 + 3i,$

$$w_1 = z_2 + 2\sqrt{2}e^{3\pi i/4} = -1 + i + 2(-1 + i) = -3 + 3i,$$
$$w_2 = z_2 + 2\sqrt{2}e^{5\pi i/4} = -1 + i + 2(-1 - i) = -3 - i,$$
$$w_3 = z_2 + 2\sqrt{2}e^{7\pi i/4} = -1 + i + 2(1 - i) = 1 - i.$$

Exercise 2:

For a function $f : D \to \mathbb{C}$ with $D \subset \mathbb{C}$ open and $z_0 \in D$, prove the following equivalence:

$$f \text{ is continuous in } z_0 \quad \Leftrightarrow \quad \mathrm{Re}(f), \mathrm{Im}(f) : D \to \mathbb{R} \text{ are continuous in } z_0 \,.$$

Solution:

Given an arbitrary sequence of numbers $(z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $\lim\limits_{n\to\infty} z_n = z_0$, then it is

$$\begin{aligned} \mathrm{Re}f(z_0) + i\,\mathrm{Im}f(z_0) &= f(z_0) = f(\lim_{n\to\infty} z_n) = \lim_{n\to\infty} f(z_n) \\ &= \lim_{n\to\infty} \mathrm{Re}f(z_n) + i \lim_{n\to\infty} \mathrm{Im}f(z_n) \end{aligned}$$

From this one gets $\mathrm{Re}f(z_0) = \lim\limits_{n\to\infty} \mathrm{Re}f(z_n)$ and $\mathrm{Im}f(z_0) = \lim\limits_{n\to\infty} \mathrm{Im}f(z_n)$.

Hand in until: 21.4.