

Complex functions for Engineering Students

Solutions of Exercise class 1

Exercise 1:

Given the complex numbers $z_1 = 3 + 2i$ and $z_2 = 5 - 4i$, determine the Cartesian representation of

a) $z_1 + z_2$, $|z_1 + z_2|$, $4z_1 - 7iz_2$, $4\bar{z}_1 - 7i\bar{z}_2$,

b) $z_1 \cdot z_2$, $\bar{z}_1 \cdot \bar{z}_2$, $z_1^3 \cdot z_2^2$, $\operatorname{Re}(z_1^3) \cdot \operatorname{Im}(z_2^2)$,

c) $\frac{z_1}{z_2}$, $\frac{\operatorname{Im}(z_1)}{\operatorname{Re}(z_2)}$.

Solution:

a) $z_1 + z_2 = 3 + 2i + 5 - 4i = 8 - 2i$

$$|z_1 + z_2| = |8 - 2i| = \sqrt{8^2 + (-2)^2} = 2\sqrt{17}$$

$$4z_1 - 7iz_2 = 4(3 + 2i) - 7i(5 - 4i) = 12 + 8i - 35i - 28 = -16 - 27i$$

$$4\bar{z}_1 - 7i\bar{z}_2 = 4(3 - 2i) - 7i(5 + 4i) = 12 - 8i - 35i + 28 = 40 - 43i$$

b) $z_1 \cdot z_2 = (3 + 2i)(5 - 4i) = 15 - 12i + 10i + 8 = 23 - 2i$

$$\bar{z}_1 \cdot \bar{z}_2 = \overline{z_1 \cdot z_2} = 23 + 2i$$

$$\begin{aligned} z_1^3 \cdot z_2^2 &= (3 + 2i)^3(5 - 4i)^2 = (-9 + 46i)(9 - 40i) \\ &= -81 + 360i + 414i + 1840 = 1759 + 774i, \end{aligned}$$

$$\operatorname{Re}(z_1^3) \cdot \operatorname{Im}(z_2^2) = -9 \cdot (-40) = 360,$$

c) $\frac{z_1}{z_2} = \frac{3 + 2i}{5 - 4i} = \frac{(3 + 2i)(5 + 4i)}{(5 - 4i)(5 + 4i)} = \frac{15 + 12i + 10i - 8}{25 + 16} = \frac{7}{41} + \frac{22}{41}i$

$$\frac{\operatorname{Im}(z_1)}{\operatorname{Re}(z_2)} = \frac{2}{5}$$

Exercise 2:

Consider the complex numbers

$$z_1 = 1, z_2 = i, z_3 = -1, z_4 = -i.$$

- a) Express $z_1 + z_2, z_2 + z_3, z_1 + z_4$ in polar coordinates.
 b) Compute in Cartesian and polar coordinates

$$(z_1 + z_2)^7, \quad \frac{z_2 + z_3}{\bar{z}_1 + \bar{z}_2}, \quad \frac{z_1 + z_4}{z_2}.$$

Solution:

- a) $z_1 + z_2 = 1 + i :$

$$r = |1 + i| = \sqrt{2}, \varphi = \arctan(1) = \frac{\pi}{4} \Rightarrow z_1 + z_2 = \sqrt{2}e^{\pi i/4}$$

$$z_2 + z_3 = -1 + i :$$

$$r = |-1 + i| = \sqrt{2},$$

$$\varphi = \pi + \arctan(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \Rightarrow z_2 + z_3 = \sqrt{2}e^{3\pi i/4}$$

$$z_1 + z_4 = 1 - i :$$

$$r = |1 - i| = \sqrt{2}, \varphi = 2\pi + \arctan(-1) = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$$\Rightarrow z_1 + z_4 = \sqrt{2}e^{7\pi i/4}$$

- b) $(z_1 + z_2)^7 = (\sqrt{2}e^{\pi i/4})^7 = 8\sqrt{2}e^{7\pi i/4} = 8(1 - i) = 8 - 8i$

$$\frac{z_2 + z_3}{\bar{z}_1 + \bar{z}_2} = \frac{\sqrt{2}e^{3\pi i/4}}{\sqrt{2}e^{-\pi i/4}} = e^{3\pi i/4 + \pi i/4} = e^{\pi i} = -1.$$

$$\frac{z_1 + z_4}{z_2} = \frac{1 - i}{i} = -1 - i$$

$$r = |-1 - i| = \sqrt{2},$$

$$\varphi = \pi + \arctan(1) = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \Rightarrow \frac{z_1 + z_4}{z_2} = \sqrt{2}e^{5\pi i/4}$$

Exercise 3:

Compute all solutions of

$$z^6 = 1$$

in polar and Cartesian coordinates.

Solution:

There are exactly 6 different roots of unity. They are determined by

$$z_k = e^{i2\pi k/6}, \quad k = 0, 1, 2, 3, 4, 5.$$

$$z_0 = e^{i2\pi \cdot 0/6} = 1,$$

$$z_1 = e^{i2\pi/6} = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i \frac{\sqrt{3}}{2},$$

$$z_2 = e^{i4\pi/6} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i \frac{\sqrt{3}}{2},$$

$$z_3 = e^{i6\pi/6} = \cos(\pi) + i \sin(\pi) = -1,$$

$$z_4 = e^{i8\pi/6} = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i \frac{\sqrt{3}}{2},$$

$$z_5 = e^{i10\pi/6} = \cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i \frac{\sqrt{3}}{2}.$$

Dates of classes: 3.4.- 7.4.