Exercise 1: (2 points)

For the exponential function exp compute the image of the set D with drawing

$$D = \left\{ z \in \mathbb{C} \,|\, 0 \leq \mathrm{Re}(z) \leq \ln(2)\,,\; 0 \leq \mathrm{Im}(z) \leq \frac{\pi}{2} \right\}.$$

Solution:

(2 points)
With $z = x + iy$ one gets $\exp(z) = e^z = e^x(\cos(y) + i\sin(y))$ and from this

$$\exp(D) = \left\{ z = re^{i\varphi} \in \mathbb{C} \,|\, 1 \leq r \leq 2\,,\; 0 \leq \varphi \leq \frac{\pi}{2} \right\}$$

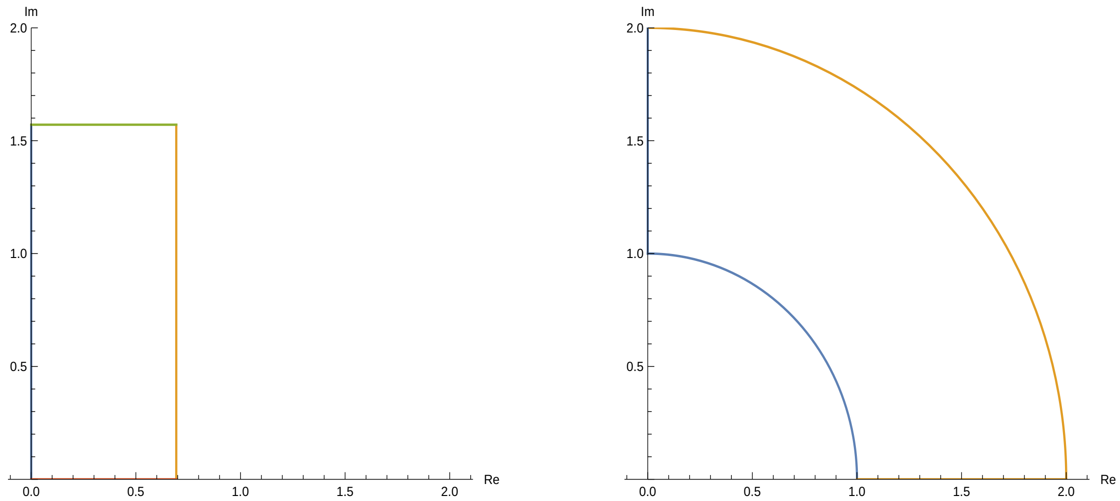


Figure 1 Sets D and $\exp(D)$

Exercise 2: (2+1+2 points)

a) Determine the two points z_1 and z_2 lying symmetrically with respect to the imaginary axis and to the circumference $K = \{z \in \mathbb{C} \mid \ |z+5| = 3\}$.

(Hint: z_1 and z_2 lie on the real axis.)

b) Compute all the Möbius transformations T such that it holds

$$T(z_1) = 0\,, \quad T(z_2) = \infty\,.$$

c) Determine the image of K under T, in case additionally it holds $T(-2) = i$.

Solution:

a) (2 points)
Points symmetrical to G and K lie on the real axis.

From the symmetry respect to G it follows $z_1 = a$ and $z_2 = -a$ with $a \in \mathbb{R}$.

The symmetry to $K = \{z \in \mathbb{C} \mid |z+5| = 3\}$ returns

$$9 = (z_1+5)(\bar{z}_2+5) = (a+5)(-a+5) = -a^2+25 \ \Rightarrow\ a = \pm 4\,.$$

b) (1 point)
All Möbius transformations are $T(z) = k \cdot \dfrac{z-z_1}{z-z_2}$.

For $z_1 = 4$ and $z_2 = -4$ for example one gets

$$T(z) = k \cdot \frac{z-4}{z+4}\,, \quad k \in \mathbb{C}\backslash\{0\}.$$

c) (2 points)
Points z_1 and z_2 lie symmetrically with respect to the circumference K and thus they are also symmetric to the image circle. This latter because of $T(z_1) = 0$ and $T(z_2) = \infty$ is a circumference around the origin with radius $R = |T(-2)| = |i| = 1$, since $z_3 = -2$ lies on K.

Exercise 3: (2+1 points)

For the function f given by

$$f(z) = z^2 - \bar{z}^2 + 2 \cdot \mathrm{Re}(z^2) + 3$$

decide (with justification) whether

a) $f(z)$ is holomorphic and

b) $\mathrm{Re}(f(z))$ is harmonic.

Solution:

a) (2 points)

f is holomorphic, because with $z = x + iy$ it holds

$$\begin{aligned} f(z) &= z^2 - \bar{z}^2 + 2 \cdot \mathrm{Re}(z^2) + 3 \\ &= x^2 - y^2 + i2xy - (x^2 - y^2 - i2xy) + 2(x^2 - y^2) + 3 \\ &= \underbrace{2(x^2 - y^2) + 3}_{=u(x,y)} + i(\underbrace{4xy}_{=v(x,y)}) \, . \end{aligned}$$

Here u, v are continuously partially differentiable and the Cauchy-Riemann differential equations

$$u_x = 4x = v_y \, , \quad v_x = 4y = -u_y$$

hold.

b) (1 point)

$u = \mathrm{Re}(f(z))$ is harmonic, because $f(z)$ is holomorphic.

(alternatively $\Delta u = 4 - 4 = 0$)

Exercise 4: (1+1 points)

Compute the line integral

a) $\displaystyle\int_c \frac{1}{z^2}\,dz$ for $c(\varphi) = e^{i\varphi}$ with $\pi \le \varphi \le \frac{3\pi}{2}$,

b) $\displaystyle\oint_{|z-i|=2} \frac{e^z}{(z+1)^5}\,dz$ with positively oriented path of $|z-i| = 2$.

Solution:

a) (1 point)

Since the quarter of circumference c runs in the holomorphic domain of $\frac{1}{z^2}$, by means of the primitive function we get

$$\int_c \frac{1}{z^2}\,dz = \int_{-1}^{-i} \frac{1}{z^2}\,dz = -\frac{1}{z}\Big|_{-1}^{-i} = \frac{1}{i} + \frac{1}{-1} = -1 - i.$$

Alternatively with $c(\varphi) = e^{i\varphi}$ and $\pi \le \varphi \le \frac{3\pi}{2}$ one finds

$$\int_c \frac{1}{z^2}\,dz = \int_{\pi}^{3\pi/2} \frac{ie^{i\varphi}}{e^{2i\varphi}}\,d\varphi = -\,e^{-i\varphi}\Big|_{\pi}^{3\pi/2} = -(i - (-1)) = -1 - i$$

b) (1 point)

Since the singularity $z_1 = -1$ lies inside the circumference $|z - i| = 2$, the generalized Cauchy integral formula returns

$$\oint_{|z-i|=2} \frac{e^z}{(z+1)^5}\,dz = 2\pi i \frac{(e^z)''''}{4!}\Big|_{z=-1} = \frac{\pi i}{12e}.$$

Exercise 5: (2+2+1+1 points)

Let the function f defined by $f(z) = \dfrac{12}{z^2+4}$.

a) Determine the type of all singularities of f and compute the corresponding residues.

b) Determine and draw the convergence domain of all power series expansions of f around $z_0 = i$.

c) Provide the complex partial fraction decomposition of f.

d) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{12}{x^2+4}\, dx$.

Solution:

a) (2 points)

$$f(z) = \frac{12}{z^2+4} = \frac{12}{(z-2i)(z+2i)}$$

Thus f has poles of the first order in $z_1 = 2i$ and $z_2 = -2i$.

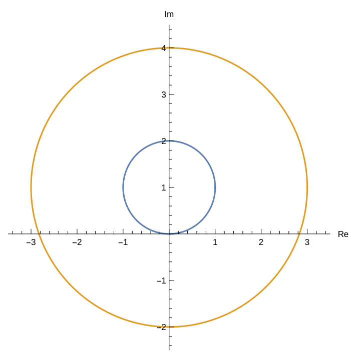
$$\mathrm{Res}(f;2i) = \left.\frac{12}{z+2i}\right|_{z=2i} = \frac{12}{2i+2i} = \frac{3}{i} = -3i,$$

$$\mathrm{Res}(f;-2i) = \left.\frac{12}{z-2i}\right|_{z=-2i} = \frac{12}{-2i-2i} = -\frac{3}{i} = 3i$$

b) (2 points)

The radii of the convergence domains are given by the distances of $z_0 = i$ from the singularities $\pm 2i$ of f

Taylor series in $|z-i| < 1$,
Laurent series for $1 < |z-i| < 3$,
Laurent series for $3 < |z-i|$.



c) (1 point)

$$\begin{aligned} f(z) &= h(z,2i) + h(z,-2i) = \frac{\mathrm{Res}(f;2i)}{z-2i} + \frac{\mathrm{Res}(f;-2i)}{z+2i} \\ &= -\frac{3i}{z-2i} + \frac{3i}{z+2i} \end{aligned}$$

d) (1 point)

$$\int_{-\infty}^{\infty} \frac{12}{x^2+4}\, dx = 2\pi i \cdot \mathrm{Res}(f;2i) = 6\pi$$

Exercise 6: (2 points)

Let the function f with $f(z) = \dfrac{\cos(z) - 1}{z^3}$ be given.

For f indicate the Laurent series development converging around $z_0 = 0$, classify all the singularities and determine the corresponding residues.

Solution:

(1 point)

$$\begin{aligned} f(z) &= \frac{\cos(z) - 1}{z^3} = \frac{1}{z^3}\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} \mp \cdots - 1\right) \\ &= -\frac{1}{2z} + \frac{z}{4!} - \frac{z^3}{6!} + \frac{z^5}{8!} \mp \cdots = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k-3} \end{aligned}$$

(1 point)

The only singularity $z_0 = 0$ is a pole of the first order with $\text{Res}(f; 0) = -\dfrac{1}{2}$.