Exercise 1: (2 points)

Determine the image of

$$K := \left\{ z \in \mathbb{C} \,|\, -\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{4} \,,\, |z| \leq 1 \right\}$$

under the mapping defined by $f(z) = z^2 + 1$ and draw it.

Solution:

(2 points)
With $f_1(z) = z^2 = (re^{i\varphi})^2 = r^2 e^{i2\varphi}$ and $f_2(w) = f_1(z) + 1$ one obtains the right semicircle around zero of radius $r = 1$

$$f_1(K) = K_1 := \left\{ z \in \mathbb{C} \,|\, -\frac{\pi}{2} \leq \arg(z) \leq \frac{\pi}{2} \,,\, |z| \leq 1 \right\}$$

and hence the right semicircle of radius $r = 1$ around $w_0 = 1$.

$$f(K) = f_2(K_1) = \left\{ w = z + 1 \in \mathbb{C} \,|\, -\frac{\pi}{2} \leq \arg(w - 1) \leq \frac{\pi}{2} \,,\, |w - 1| \leq 1 \right\}$$

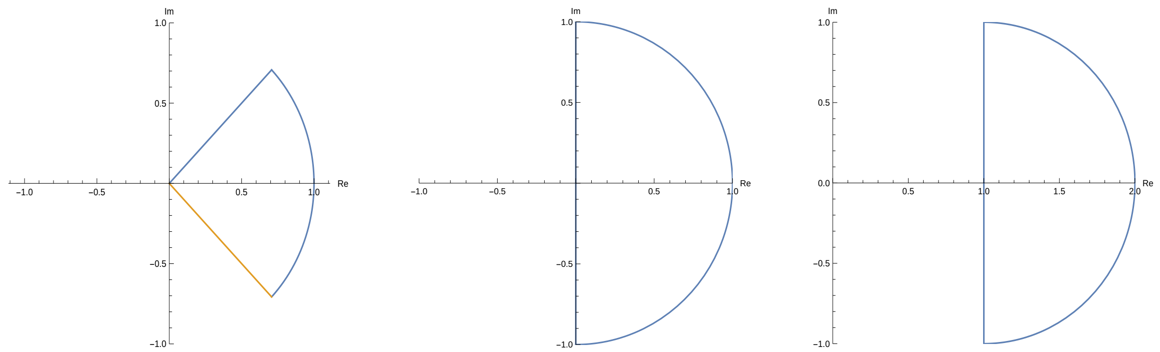

Figure 1 Sets $K,\ f_1(K),\ f(K)$

Exercise 2: (2+1+2 points)

Let a Möbius transformation $w = T(z)$ be given with

$$T(-2i) = 0 \quad \text{and} \quad T(0) = -3.$$

a) Determine T such that the lower half plane $\text{Im}(z) \leq 0$ is mapped onto the circular disk
$$K := \{w \in \mathbb{C} \mid |w| \leq R\}.$$
(Hint: $z_1 = -2i$ and $z_2 = 2i$ lie symmetrically with respect to the real axis.)

b) Compute the radius R of the disk K.

c) Calculate T.

Solution:

a) (2 points)

$z_1 = -2i$ and $z_2 = 2i$ lie symmetrically to $\mathbb{R}$. Thus $w_1 = T(z_1)$ and $w_2 = T(z_2)$ are symmetric respect to the image of the real axis. From the choice of $w_2 = T(z_2) = \infty$ the real axis is mapped onto a circle around $w_1 = T(-2i) = 0$.

Since $z_1 = -2i$ lies in the lower half plane, this will be mapped by T onto the circular disk
$$K := \{w \in \mathbb{C} \mid |w| \leq R\}.$$

b) (1 point)

Since it holds $0 \in \mathbb{R}$, $T(0) = -3$ lies on the image circle. Thus it is $R = |T(0)| = 3$.

c) (2 points)

The conditions $T(-2i) = 0$ and $T(2i) = \infty$ are satisfied by
$$T(z) = k\frac{z+2i}{z-2i}.$$

With $T(0) = -3$ we get
$$T(0) = k\frac{2i}{-2i} = -3 \quad \Rightarrow \quad k = 3 \quad \Rightarrow \quad w = T(z) = \frac{3(z+2i)}{z-2i}.$$

Alternatively, from the three-points formula with $z_1 = -2i, z_2 = 2i$ and $z_3 = 0$ as well as $w_1 = 0, w_2 = \infty$ and $w_3 = -3$ one obtains
$$\frac{z+2i}{z-2i} : \frac{2i}{-2i} = \frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2} = \frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = \frac{w-0}{-3-0} \Rightarrow w = \frac{3(z+2i)}{z-2i}.$$

Exercise 3: (1+2 points)

Let the function defined by $u(x,y) = y^2 + 3x - x^2 + e^{-x}\cos(y)$.

a) Show that u is harmonic.

b) Construct a function $v(x,y)$ such that the function $f(z) = u(x,y) + iv(x,y)$ with $z = x + iy$ is holomorphic.

Solution:

a) (1 point)

$$\begin{aligned} \Delta u &= (y^2 + 3x - x^2 + e^{-x}\cos(y))_{xx} + (y^2 + 3x - x^2 + e^{-x}\cos(y))_{yy} \\ &= -2 + e^{-x}\cos(y) + 2 - e^{-x}\cos(y) = 0 \end{aligned}$$

b) (2 points)

For the holomorphism of $f(z) = u(x,y) + iv(x,y)$ in $\mathbb{C}$ the Cauchy-Riemann differential equations must be fulfilled:

$$v_y \stackrel{!}{=} u_x = (y^2 + 3x - x^2 + e^{-x}\cos(y))_x = 3 - 2x - e^{-x}\cos(y)$$

$$\Rightarrow \quad v = 3y - 2xy - e^{-x}\sin(y) + c(x)$$

$$\Rightarrow \quad v_x = -2y + e^{-x}\sin(y) + c'(x)$$

$$\stackrel{!}{=} -u_y = -(y^2 + 3x - x^2 + e^{-x}\cos(y))_y = -2y + e^{-x}\sin(y)$$

$$\Rightarrow \quad c'(x) = 0 \quad \Rightarrow \quad c(x) = K\,, \quad K \in \mathbb{R}$$

$$\Rightarrow \quad v(x,y) = 3y - 2xy - e^{-x}\sin(y) + K$$

Exercise 4: (1+1 points)

Compute the line integrals

a) $\int\limits_c \frac{1}{z}\,dz$ for $c(\varphi) = e^{i\varphi}$ with $0 \le \varphi \le \frac{\pi}{2}$,

b) $\oint\limits_{|z-1|=1} \frac{\sin(z)}{\left(z-\frac{\pi}{2}\right)^3}\,dz$ with positively oriented path of $|z-1| = 1$.

Solution:

a) (1 point)

Since the quarter of circumference c runs in the holomorphic domain of the principal value of $\ln(z)$, by means of the primitive function we get

$$\int\limits_c \frac{1}{z}\,dz \;=\; \int\limits_1^i \frac{1}{z}\,dz = \ln(z)\big|_1^i = \ln|i| + i\frac{\pi}{2} - (\ln|1| + i\cdot 0) = \frac{\pi i}{2}.$$

Alternatively with $c(\varphi) = e^{i\varphi}$ and $0 \le \varphi \le \frac{\pi}{2}$ one finds

$$\int\limits_c \frac{1}{z}\,dz \;=\; \int\limits_0^{\pi/2} \frac{ie^{i\varphi}}{e^{i\varphi}}\,d\varphi \;=\; i\varphi\big|_0^{\pi/2} = \frac{\pi i}{2}$$

b) (1 point)

Since the singularity $z_1 = \frac{\pi}{2}$ lies inside the circumference $|z-1| = 1$, the generalized Cauchy integral formula returns

$$\oint\limits_{|z-1|=1} \frac{\sin z}{\left(z-\frac{\pi}{2}\right)^3}\,dz = 2\pi i \frac{(\sin z)''}{2!}\bigg|_{z=\frac{\pi}{2}} = -\pi i.$$

Exercise 5: (3+2+1 points)

Let the function f defined by $f(z) = \dfrac{3}{z-2} + \dfrac{6}{z-5}$.

a) For the development point $z_0 = 2$ one compute all the power series expansions of f and draw their convergence domains.

b) Determine the type of all singularities of f and give the corresponding residues.

c) Compute $\displaystyle\oint_{|z-3|=3} f(z)\,dz$ for the simple curve $|z-3| = 3$ running in the positive mathematical orientation.

Solution:

a) (3 points)

$0 < |z-2| < 3:$

$$\frac{6}{z-5} = -\frac{6}{3-(z-2)}$$

$$= -\frac{2}{1-(z-2)/3} = -2\sum_{k=0}^{\infty}\frac{(z-2)^k}{3^k}$$

$$f(z) = \frac{3}{z-2} - 2\sum_{k=0}^{\infty}\frac{(z-2)^k}{3^k}$$

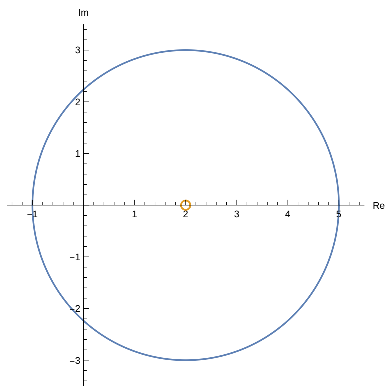

Figure 5: dotted circular disk $0 < |z-2| < 3$ and outer domain $3 < |z-2|$

$|z-2| > 3:$

$$\frac{6}{z-5} = \frac{6}{z-2-3} = \frac{3}{z-2}\cdot\frac{2}{1-3/(z-2)} = 2\sum_{k=0}^{\infty}\frac{3^{k+1}}{(z-2)^{k+1}}$$

$$f(z) = \frac{3}{z-2} + 2\sum_{k=0}^{\infty}\frac{3^{k+1}}{(z-2)^{k+1}} = \frac{9}{z-2} + 2\sum_{k=1}^{\infty}\frac{3^{k+1}}{(z-2)^{k+1}}$$

b) (2 points)

f has poles of the first order in $z_1 = 2$ and $z_2 = 5$

$\mathrm{Res}(f;2) = (z-2)f(z)|_{z=2} = 3,$

$\mathrm{Res}(f;5) = (z-5)f(z)|_{z=5} = 6.$

c) (1 point)

$$\oint_{|z-3|=3} f(z)\,dz = 2\pi i\cdot(\mathrm{Res}(f;2) + \mathrm{Res}(f;5)) = 18\pi i$$

Exercise 6: (2 points)

Let the function f be given with $f(z) = (z-2)^3 \exp\left(\frac{1}{z-2}\right)$.

For f determine the convergent Laurent series expansion around $z_0 = 2$, classify all the singularities and determine the corresponding residues.

Solution:

(1 point)

$$\begin{aligned} f(z) &= (z-2)^3 \exp\left(\frac{1}{z-2}\right) = (z-2)^3 \sum_{k=0}^{\infty} \frac{(z-2)^{-k}}{k!} = \sum_{k=0}^{\infty} \frac{(z-2)^{3-k}}{k!} \\ &= (z-2)^3 + (z-2)^2 + \frac{z-2}{2} + \frac{1}{3!} + \frac{1}{4!(z-2)} + \frac{1}{5!(z-2)^2} + \frac{1}{6!(z-2)^3} + \cdots \end{aligned}$$

(1 point)

$z_0 = 2$ is an essential singularity with $\mathrm{Res}(f;2) = \frac{1}{4!}$.