Complex functions for engineering study programs

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Sommersemester 2022

based on slides of Prof. Jens Struckmeier from Sommersemester 2021

Content of the lecture on complex functions.

1. Complex functions of a single variable.
2. Möbius–transformation.
3. Complex differentiation.
4. Conformal mappings.
5. Complex integration.
6. Cauchy's intergal formula and applicatons.
7. Taylor– and Laurent–series.
8. Isolated singularities and residue.
9. Residue.
10. Fourier–transform and partial differential equations.

Starting point: consider the cubic equation

$$x^3 = 3px + 2q$$

and the solution formula (by Gerolamo Cardano, 16th century)

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

Rafael Bombelli (also 16th century) considers the equation

$$x^3 = 15x + 4$$

and obtains the solution formula

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Bombelli defines the imagnary unit i via $i^2 = -1$, the complex numbers and their summation and multiplication.

First ideas to introduce the complex numbers.

Starting point: Use the symbolic solution i for the equation $x^2 + 1 = 0$, such that

$$i^2 = -1$$

The "number" i is called imaginary unit.
Next step: With the imaginary unit we build the set of numbers

$$\mathbb{C} = \{a + ib \,|\, a, b \in \mathbb{R}\}$$

Then we introduce the following rules on $\mathbb{C}$:

- **Addition**

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

- **Multiplication**

$$(a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

With this $\mathbb{C}$ obtaines an algebraic structure.

Fundamental question about the complex numbers.

- What exactely is i?
- With the above rules can we "calculate" without contradictions?
- Are the above rules consistent with the related rules in $\mathbb{R}$?
- Can we order the complex numbers?
- Is there alternative representations of the complex numbers?
- Is there a geometric interpretation of the operations in $\mathbb{C}$?
- ...
- Why do we introduce the complex numbers?
- ... and later complex functions?
- Is there interesting applications of the complex numbers in eingineering?

On the contruction of the complex numbers.

Starting point: consider the set $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}$ with **addition**

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

and **multiplication**

$$(a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \qquad \text{for } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

Observation: The multiplication is associative and commutative; in addition we have

$$(a,b) \cdot (1,0) = (a,b) \qquad \text{for } (a,b) \in \mathbb{R}^2,$$

i.e. $(1,0) \in \mathbb{C}$ is neutral element of the multiplication. The equation

$$(a,b) \cdot (x,y) = (1,0) \qquad \text{for } (a,b) \neq (0,0)$$

has the unique solution, the multiplicative inverse to (a,b),

$$(x,y) = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right)$$

On the structure of the complex numbers.

Remark: The set $\mathbb{R}^2$ forms together with the addition and the multiplication a field, the field of complex numbers which we denote by $\mathbb{C}$.

Observation: the map $\varphi : \mathbb{R} \to \mathbb{C}$, defined by $\varphi(a) = (a, 0)$ is injectiv. For all $a_1, a_2 \in \mathbb{R}$ we have

$$\begin{aligned} \varphi(a_1 + a_2) &= (a_1 + a_2, 0) = (a_1, 0) + (a_2, 0) = \varphi(a_1) + \varphi(a_2) \\ \varphi(a_1 a_2) &= (a_1 a_2, 0) = (a_1, 0) \cdot (a_2, 0) = \varphi(a_1) \cdot \varphi(a_2) \end{aligned}$$

Conclusion:

- We can identify the real numbers as complex numbers of the form $(a, 0)$;
- The real numbers form a subfield of $\mathbb{C}$;
- The rules for calculation in $\mathbb{C}$ are consistent with the rules in $\mathbb{R}$.

The field of real numbers is ordered.

Remark: The real numbers form a ordered filed; the following order axioms hold.

- For every $x \in \mathbb{R}$ it is $x > 0$ or $x = 0$ or $x < 0$;
- For $x > 0$ and $y > 0$ it is $x + y > 0$;
- For $x > 0$ and $y > 0$ it is $xy > 0$.

Question: Is the field of complex numbers $\mathbb{C}$ ordered?

Answer: **NO!**

In an ordered field nonzero square numbers are positiv. If $\mathbb{C}$ would be ordered then

$$0 < 1^2 = 1 \qquad \text{and} \qquad 0 < i^2 = -1$$

the contradiction $0 < 1 + (-1) = 0$.

A simpler notation for the complex numbers.

Simplification of the notation:

- For $a \in \mathbb{R}$ we write a instead of $(a, 0)$;
- We denote the complex unit $(0, 1)$ by i;
- With this every complex number (a, b) can be written

$$(a, b) = (a, 0) + (0, b) \cdot (0, 1) = a + b \cdot i = a + ib$$

and is is

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (-1, 0) = -1.$$

Conclusion: We have constructed a field $\mathbb{C}$ which includes $\mathbb{R}$. The equation

$$x^2 + 1 = 0$$

is solvable in $\mathbb{C}$. The only two solutions are $\pm i$.

Real and imaginary part.

From now on we denote complex numbers by z or w. For

$$z = x + iy \in \mathbb{C} \qquad \text{for } x, y \in \mathbb{R}$$

x is called the real part and y is called the imaginary part of z, shortly

$$x = \mathrm{Re}(z) \qquad \text{and} \qquad y = \mathrm{Im}(z)$$

We have the following rules

$$\begin{aligned}
\mathrm{Re}(z+w) &= \mathrm{Re}(z) + \mathrm{Re}(w) && \text{for } z, w \in \mathbb{C} \\
\mathrm{Im}(z+w) &= \mathrm{Im}(z) + \mathrm{Im}(w) && \text{for } z, w \in \mathbb{C} \\
\mathrm{Re}\,(az) &= a\,\mathrm{Re}(z) && \text{for } z \in \mathbb{C}, a \in \mathbb{R} \\
\mathrm{Im}\,(az) &= a\,\mathrm{Im}(z) && \text{for } z \in \mathbb{C}, a \in \mathbb{R}
\end{aligned}$$

and

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2} \qquad \text{for } z \neq 0.$$

The complex plane.

Geometric rappresentation:

We identify $z = (x, y) \in \mathbb{C}$ as point in the

complex plane (Gauß plane)

given by the cartesian coordinate system of the $\mathbb{R}^2$, with a real axis, $\mathbb{R}$, and an imaginary axis, $i \cdot \mathbb{R}$.

Geometric rappresentation of the addition:

The usual addition of vectors according to the parallelogram rule.

Rappresentation of the addition of two complex numbers on **slide**.

Conjugation of complex numbers.

We obtain for every complex number $z = x + iy$ by mirrowing along the real axis a complex number

$$\bar{z} = x - iy \in \mathbb{C}$$

the conjugate complex number.

We have the following rules

$$\begin{aligned}
\overline{z+w} &= \bar{z} + \bar{w} && \text{for } z, w \in \mathbb{C} \\
\overline{zw} &= \bar{z} \cdot \bar{w} && \text{for } z, w \in \mathbb{C} \\
\overline{(\bar{z})} &= z && \text{for } z \in \mathbb{C} \\
z\bar{z} &= x^2 + y^2 && \text{for } z = x + iy \in \mathbb{C} \\
\mathrm{Re}(z) &= (z + \bar{z})/2 && \text{for } z \in \mathbb{C} \\
\mathrm{Im}(z) &= (z - \bar{z})/2i && \text{for } z \in \mathbb{C}
\end{aligned}$$

In particular it holds $z = \bar{z}$ if an only if $z \in \mathbb{R}$.

The absolute value.

We set

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} \qquad \text{for } z = x + iy \in \mathbb{C}$$

for the absolute value of z and $|z - w|$ for the distance of two numbers $z, w \in \mathbb{C}$ in the complex plane.

- Then $|z| = |z - 0|$ represents the Eucledian distance of z to the origin.
- For $z \in \mathbb{R}$ the absolute value $|z|$ coincides with the usual absolute value for real numbers.
- We have the following estimates.

$$-|z| \leq \mathrm{Re}(z) \leq |z| \qquad \text{and} \qquad -|z| \leq \mathrm{Im}(z) \leq |z| \qquad \text{for } z \in \mathbb{C}$$

Theorem: The absolute value defines a norm on $\mathbb{C}$, since we have the relations

1. $|z| \geq 0$ for all $z \in \mathbb{C}$ and $|z| = 0$ if and only if $z = 0$;
2. $|z + w| \leq |z| + |w|$ for all $z, w \in \mathbb{C}$ (triangle inequality);
3. $|zw| = |z| \cdot |w|$ for all $z, w \in \mathbb{C}$.

The Euler's formula.

In the complex plane we have for $z = x + iy$ using polar coordinates

$$(x, y) = |z|(\cos(\varphi), \sin(\varphi))$$

the Euler's formula

$$z = |z| \exp(i\varphi) = |z|(\cos(\varphi) + i \sin(\varphi))$$

where $\varphi \in [0, 2\pi)$ for $z \neq 0$ represents the (unique) angle between the positive real axis and the ray from 0 through $z = (x, y)$.

The angle $\varphi \in [0, 2\pi)$ is called polar angle (azimuth, argument) of $z \neq 0$, shortly

$$\varphi = \arg(z) \in [0, 2\pi)$$

Example: $i = (0, 1) = \exp(i\pi/2),\ -1 = i^2 = \exp(i\pi)$, thus $e^{i\pi} + 1 = 0$.

The geometry of multiplikation and division.

Using polar coordinates the multiplication of two complex numbers $z, w \in \mathbb{C}$ can be interpreted as rotational dilation in the complex plane, since for

$$z = |z|\,(\cos(\varphi), \sin(\varphi)) \qquad \text{and} \qquad w = |w|\,(\cos(\psi), \sin(\psi))$$

we have

$$\begin{aligned} z \cdot w &= |z| \cdot |w|\,(\cos(\varphi) + i\sin(\varphi)(\cos(\psi) + i\sin(\psi)) \\ &= |z| \cdot |w|\,(\cos(\varphi+\psi) + i\sin(\varphi+\psi)) = |z| \cdot |w|\,\exp(i(\varphi+\psi)) \end{aligned}$$

and with the Euler's formula

$$z \cdot w = |z| \cdot |w|\,\exp(i\varphi)\,\exp(i\psi) = |z| \cdot |w|\,\exp(i(\varphi+\psi))$$

For the division of two complex numbers $z, w \in \mathbb{C}$ with $z \neq 0$ we have in analogy

$$\frac{z}{w} = \frac{|z|}{|w|}\,\exp(i(\varphi-\psi)) = \frac{|z|}{|w|}\,(\cos(\varphi-\psi) + i\sin(\varphi-\psi))$$

Powers and roots of unity.

For the n–th power z^n of $z \in \mathbb{C}$ we have

$$z^n = (|z| \exp(i\varphi))^n = |z|^n \exp(in\varphi) = |z|^n (\cos(n\varphi) + i \sin(n\varphi))$$

The equation

$$z^n = 1$$

has n pairwise different solutions

$$z_k = \exp\left(i \frac{2\pi k}{n}\right) \qquad \text{for } k = 0, \ldots, n-1.$$

These solutions are called n–th roots of unity.

A complex function $w = f(z)$ is a map $f : D \to \mathbb{C}$ with $D \subset \mathbb{C}$, i.e. for every $z \in D$ there is a unique $w = f(z) \in \mathbb{C}$.

The set D is the domain (of defintion) of f. The set

$$W = f(D) = \{f(z) \,|\, z \in D\}$$

is called the codomain.

Notation:

$$\begin{aligned} z &= x + iy \\ w &= u + iv \\ u &= u(x, y) = \mathrm{Re}(w) \\ v &= v(x, y) = \mathrm{Im}(w) \end{aligned}$$

For a geometric representation of complex functions often images of coordinate nets are used.

2.1 Linear functions

Definition: A complex function f is called linear, if f for fixed complex constants $a, b \in \mathbb{C}, a \neq 0$, has a representation of the following form

$$f(z) = az + b \qquad \text{for } z \in \mathbb{C}.$$

Question: Can we interpet linear functions geometrically?

Special case 1: The choice $a = 1$ leads to a translation of b,

$$f(z) = z + b \qquad \text{for } z \in \mathbb{C}$$

Special case 2: The choice $a \in (0, \infty)$ and $b = 0$ leads to a dilation or contraction,

$$f(z) = az \qquad \text{for } z \in \mathbb{C},$$

i.e. the absolute value of z is dilated $(a > 1)$ or contracted $(0 < a < 1)$. In general we talk about a scaling with scaling factor $a > 0$.

Other special cases of linear functions.

Special case 3: The choice $a \in \mathbb{C}$ with $|a| = 1$ and $b = 0$ leads to a rotation,

$$f(z) = az \qquad \text{for } z \in \mathbb{C},$$

More precisely: a rotation with angle $\alpha \in [0, 2\pi)$, where $\alpha = \arg(a)$ and $a = \exp(i\alpha)$.

Special case 4: The choice $a \in \mathbb{C}$, $a \neq 0$ and $b = 0$ leads to a rotational dilation

$$f(z) = az \qquad \text{for } z \in \mathbb{C},$$

which we understand as a combination of a rotation and a scaling.

More precisely: For

$$a = |a| \exp(i\alpha) \qquad \text{with } \alpha = \arg(a)$$

we have a rotation with angle $\alpha \in [0, 2\pi)$ and a scaling with factor $|a|$.

The general case of linear functions.

For $a, b \in \mathbb{C}$, $a \neq 0$, every linear function

$$f(z) = az + b = |a| \exp(i\alpha)z + b$$

can be written as composition

$$f = f_3 \circ f_2 \circ f_1$$

of three maps,

1. $f_1(z) = \exp(i\alpha)z$ a rotation with angle $\alpha = [0, 2\pi)$;
2. $f_2(z) = |a|z$ a dilation with scaling factor $|a| > 0$;
3. $f_3(z) = z + b$ a shift with a vector b.

Remark: rotation f_1 and dilation f_2 commute, i.e. can be exchanged since

$$f_2 \circ f_1 = f_1 \circ f_2$$

and thus

$$f = f_3 \circ f_2 \circ f_1 = f_3 \circ f_1 \circ f_2$$

2.2 Quadratic functions

Definition: A complex function f is called quadratic, if f for fixed constants $a, b, c \in \mathbb{C}$, $a \neq 0$, has the following form.

$$f(z) = az^2 + bz + c \qquad \text{for} \quad z \in \mathbb{C}$$

First we consider the geometric behaviour of the function

$$f(z) = z^2 \qquad \text{for} \quad z \in \mathbb{C}$$

To do so we consider the image under f of straight lines parallel to the coordinate axes.

Set $w = z^2$. Then with $z = x + iy$ and $w = u + iv$ we obtain the representation

$$w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

and thus

$$u = x^2 - y^2 \qquad \text{and} \qquad v = 2xy.$$

Images of straight lines parallel to the axes under $z \mapsto z^2$.

For the image of a straight line $y = y_0$ parallel to the x-axis we obtain

$$u = x^2 - y_0^2 \qquad \text{and} \qquad v = 2xy_0$$

For $y_0 = 0$ (the x-axis) we obtain $u = x^2$ and $v = 0$.

For $y_0 \neq 0$ we can eliminate x with $x = v/(2y_0)$ and obtain

$$u = \frac{v^2}{4y_0^2} - y_0^2,$$

a parabola open to the right, symmetric with respect to the u-axes with focus in zero, intersecting the u-axis in $u = -y_0^2$ and the v-axis in $v = \pm 2y_0^2$.

Conclusion: The family of straight lines parallel to the x-axis by the quadratic function $f(z) = z^2$ is mapped on a family of confocal (i.e. same symmetry axis, same focus) parabolas, open to the right.

The lines $y = y_0$ and $y = -y_0$ are mapped onto the same parabola.

Images of straight lines parallel to the axes under $z \mapsto z^2$.

For the image of a straight line $x = x_0$ parallel to the y-axis we obtain

$$u = x_0^2 - y^2 \qquad \text{und} \qquad v = 2x_0 y$$

For $x_0 = 0$ (the y-axis) we obtain $u = -y^2$ and $v = 0$.

For $x_0 \neq 0$ we can eliminate y with $y = v/(2x_0)$ and obtain

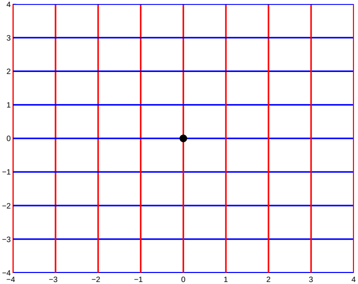
$$u = x_0^2 - \frac{v^2}{4x_0^2}$$

a parabola open to the left, symmetric to the u-axis with focus zero, intersecting the u-axis in $u = x_0^2$ and the v-axis in $v = \pm 2x_0^2$.

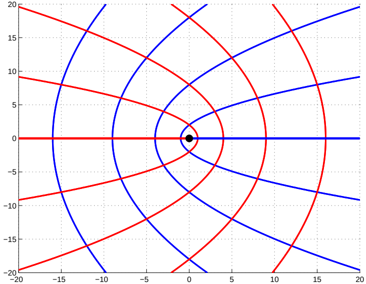
Conclusion: The family of straight lines parallel to the y-axis by the quadratic function $f(z) = z^2$ is mapped on a family of confocal parabolas, open to the left.

The lines $x = x_0$ and $x = -x_0$ are mapped onto the same parabola.

Images of straight lines parallel to the axes under $z \mapsto z^2$.



Domain.



Codomain of $f(z) = z^2$.

General quadratic functions.

For $a, b, c \in \mathbb{C}$, $a, b \neq 0$, and the representation

$$f(z) = az^2 + bz + c = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

every quadratic function can be written as a composition of 4 maps

$$f = f_4 \circ f_3 \circ f_2 \circ f_1$$

consisting in:

1. a shift $f_1(z) = z + \frac{b}{2a}$;
2. a quadratic function $f_2(z) = z^2$;
3. a rotational dilation $f_3(z) = az$;
4. a shift $f_4(z) = z - \frac{b^2}{4a} + c$.

2.3 The exponential function

Definition: The complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is defined as

$$\exp(z) = e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)) \qquad \text{for } z = x + iy.$$

We observe: The rule for the addition holds

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \qquad \text{for } z_1, z_2 \in \mathbb{C}.$$

Question: How does the complex exponential function $z \to \exp(z)$ look like?

For $w = \exp(z)$, $z = x + iy$ and $w = u + iv$ we obtain

$$w = u + iv = e^z = e^x(\cos(y) + i\sin(y))$$

and thus

$$u = e^x\cos(y) \qquad \text{and} \qquad v = e^x\sin(y)$$

Images of straight lines parallel to the axes under $z \mapsto \exp(z)$.

For the image of a straight line $y = y_0$ parallel to the x-axis we obtain

$$u = e^x \cos(y_0) \qquad \text{and} \qquad v = e^x \sin(y_0)$$

- For fixed y_0 this gives a ray starting from the origin with angle y_0 with respect to the the x-axis.
- For angles y_0 and y_1, which differ by a multiple of 2π, i.e.

  $$y_1 = y_0 + 2\pi k \qquad \text{for a } k \in \mathbb{Z},$$

  we obtain the same ray.
- **More precisely:** Due to the periodicity of $\exp(z)$ we have

  $$e^{z+2\pi ik} = e^z e^{2\pi ik} = e^z(\cos(2\pi k) + i\sin(2\pi k)) = e^z \cdot 1 = e^z.$$

  i.e. two points with identical real part, which imaginary parts only differ by a multiple of 2π, are mapped onto the same point.

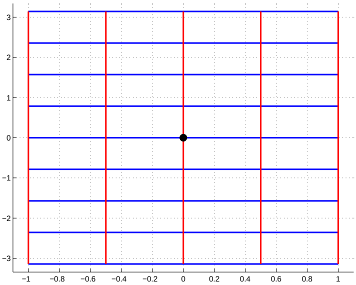
Images of straight lines parallel to the axes under $z \mapsto \exp(z)$.

For the image of a straight line $x = x_0$ parallel to the y-axis we obtain

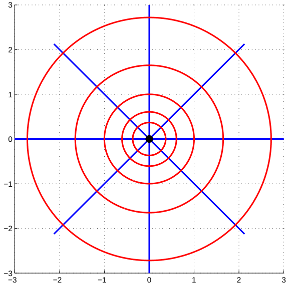
$$u = e^{x_0} \cos(y) \qquad \text{und} \qquad v = e^{x_0} \sin(y)$$

- For fixed x_0 this gives a circle around the origin with radius e^{x_0}.
- **Observe:** The origin does not lie in the codomain of the exponential function, i.e. there is no $z \in \mathbb{C}$ with $\exp(z) = 0$. Therefore $e^z \neq 0$ for all $z \in \mathbb{C}$.
- **Observation:** The exponential function maps rectangular lattices in the cartesian coordinate system onto lattices of curves which intersect orthogonally.
- **More precisely:** Curves which intersect orthogonally in the cartesian coordinate system, are mapped by the exponential function exp onto curves, which intersect orthogonally (in the images of the interesction point)
- **Even more general:** The exponential function is isogonal or conformal in $\mathbb{C} \setminus \{0\}$. More details later.

Images of straight lines parallel to the axes under $z \mapsto \exp(z)$.



Domain.



Codomain of $f(z) = \exp(z)$.

2.4 The inverse function

Definition: A complex function $f = f(z)$ is called injective, if for every point $w \in \mathbb{C}$ in the domain there is exactely one point $z \in \mathbb{C}$ in the codomain with $f(z) = w$.

Remark: A non-injective function might become injective if the domain is appropriately restricted.

Examples.

1. the linear function $f(z) = az + b$, $a \neq 0$ is injective.
2. the quadratic function $f(z) = z^2$ is not injective, since we have $f(z) = f(-z)$ for all $z \in \mathbb{C}$.
3. the complex exponential function $\exp(z)$ is not injective, since we have $\exp(z) = \exp(z + 2\pi ik)$ for all $k \in \mathbb{Z}$ and all $z \in \mathbb{C}$.

Restriction of the domain.

Remark: A non-injective function might become injective if the domain is appropriately restricted.

Example: Consider the quadratic function

$$f(z) = z^2 \qquad \text{for } z \in \mathbb{C} \text{ with } \operatorname{Re}(z) > 0$$

on the right halfplane $\{z \in \mathbb{C} \,|\, \operatorname{Re}(z) > 0\}$. There f is injective.

In this case the codomain is given by the "partly cutted" complex plane

$$\begin{aligned} \mathbb{C}^- &= \{z \in \mathbb{C} \,|\, \operatorname{Im}(z) \neq 0 \text{ or } \operatorname{Re}(z) > 0\} \\ &= \mathbb{C} \setminus \{z \in \mathbb{R} \,|\, z \leq 0\} \end{aligned}$$

Graphical representation of the domain and codomain on a slight.

The inverse function.

Definition: Let f be an injective function with domain $D(f)$ and codomain $W(f)$. Then the inverse function $f^{-1} : W(f) \to D(f)$ to f is the function, which maps every point $w \in W(f)$ onto the (unique) point $z \in D(f)$ with $f(z) = w$, i.e. it is $f^{-1}(w) = z$ and

$$\begin{aligned} (f^{-1} \circ f)(z) &= z \qquad \text{for all } z \in D(f) \\ (f \circ f^{-1})(w) &= w \qquad \text{for all } w \in W(f) \end{aligned}$$

Example: For the domain

$$D(f) = \{z = re^{i\varphi} \in \mathbb{C} \,|\, r > 0 \text{ and } -\pi/2 < \varphi < \pi/2\}$$

the exists an inverse function f^{-1} of $f(z) = z^2$ with codomain $W(f) = \mathbb{C}^-$.

For the main value uf the root $f^{-1} : W(f) \to D(f)$ it is

$$w = f^{-1}(z) = \sqrt{r}e^{i\varphi/2} \qquad \text{for } z = re^{i\varphi} \text{ with } \varphi = \arg(z) \in (-\pi, \pi).$$

2.5 The complex logarithm

Aim: To inverte the complex exponential function

$$f(z) = \exp(z).$$

Observe: The exponential function $\exp(z)$ is defined for all $z \in \mathbb{C}$ and we have

$$D(\exp) = \mathbb{C} \qquad \text{and} \qquad W(f) = \mathbb{C} \setminus \{0\}$$

for the domain and the codomain.

But: The exponential function is not injective on $\mathbb{C}$.

Also: For the construction of the inverse function $\exp^{-1}$ of exp we need to restrict the domain of exp appropriately.

Question: Let $z = x + iy \in W(\exp)$. Which values $w = u + iv$ are possible such that

$$e^w = z?$$

Construction of the complex logarithm.

Starting point: For $z = x + iy \in W(\exp)$ it should be

$$e^w = z \qquad \text{for a } w = u + iv \in \mathbb{C}.$$

Then

$$|e^w| = |e^u| = |z|$$

and thus $u = \ln(|z|)$, where $\ln : (0, \infty) \to \mathbb{R}$ denotes the real logarithm.
In addition we have

$$\arg(e^w) = \arg(e^{u+iv}) = \arg(e^u e^{iv}) = v$$

and thus $v = \arg(z) + 2\pi k$ for a $k \in \mathbb{Z}$.
Therefore the set of solutions of $e^w = z$ consists of complex numbers

$$w = \ln(|z|) + i(\arg(z) + 2\pi k) \qquad \text{with a } k \in \mathbb{Z}.$$

The set of solutions of $e^w = z$ is called complex logarithm of z.

Examples.

The function $\mathrm{Log}(z)$ denotes the complex logarithm of z.

Example 1: How does the set $\mathrm{Log}(-1)$ look like? We have $\ln(|-1|) = \ln(1) = 0$ and the argument of -1 is $\arg(-1) = \pi$. Thus

$$\mathrm{Log}(-1) = \{i(2k+1)\pi \mid k \in \mathbb{Z}\}$$

for the values of the logarithm of -1.

Example 2: How does the set $\mathrm{Log}(-1+i)$ look like? We have $|-1+i| = \sqrt{2}$ and it is $\arg(-1+i) = \frac{3\pi}{4}$ the argument of $-1+i$. Thus

$$\mathrm{Log}(-1+i) = \left\{ \ln(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi k\right) \,\middle|\, k \in \mathbb{Z} \right\}$$

for the values of the logarithm of $-1+i$.

Example 3: For $x > 0$ it is $\mathrm{Log}(x) = \{\ln(x) + 2\pi i k \mid k \in \mathbb{Z}\}$.

The principal value of the logarithm.

The previous consoderations for the equation

$$z = e^w$$

show that the exponantial function is injective on the strip

$$S = \{w \in \mathbb{C} \mid -\pi < \operatorname{Im}(w) < \pi\}.$$

The related codomain is $\mathbb{C}^-$.
The unique value of $\operatorname{Log}(z)$ being element in the strip S is

$$w = \log(|z|) + i \arg(z) \qquad \text{with } -\pi < \arg(z) < \pi.$$

This value is called principal value of the logarithm of z, shortly $\ln(z)$.

Remark: The principal value is only defined in the "opened" complex plane $\mathbb{C}^-$. On the negative real axis and at $z = 0$ the $\ln(z)$ is not defined. On the positive real axis $\ln(z)$ coincides with the real logarithm $\ln(x)$.

2.6 The Joukowski–function

The Joukowski–function is defined as

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{for } z \neq 0,$$

and has an interesting connection to fluid mechanics.

Observation: We have the symmetry

$$f(z) = f(1/z) \qquad \text{for } z \neq 0.$$

Aim: Analyse the geometric behaviour of the Joukowski–function.

To do so determine for

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

the images of the circles $|z| = \mathrm{const.}$ and the rays $\arg(z) = \mathrm{const.}$.

Geometric behaviour of the Joukowski–function.

For $z = re^{i\varphi}$ and $w = u + iv$ we obtain

$$u + iv = \frac{1}{2}\left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right)$$

and thus

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\varphi) \qquad \text{and} \qquad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\varphi).$$

For the images of the circles $r \equiv r_0 > 0$ we obtain the parameterized form

$$\left.\begin{array}{rcl} u & = & \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right)\cos(\varphi) \\ v & = & \frac{1}{2}\left(r_0 - \frac{1}{r_0}\right)\sin(\varphi) \end{array}\right\} \qquad 0 \le \varphi < 2\pi.$$

For the unit circle $r_0 \equiv 1$ we have $u = \cos(\varphi)$, for $0 \le \varphi < 2\pi$, and $v \equiv 0$, i.e. the line between -1 and 1, which is reached twice.

Geometric behaviour of the Joukowski–function.

For $r_0 \neq 1$ we can eliminate φ and we obtain the ellipse

$$\frac{u^2}{\frac{1}{4}\left(r_0 + \frac{1}{r_0}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r_0 - \frac{1}{r_0}\right)^2} = 1$$

with the semi axes

$$a = \frac{1}{2}\left(r_0 + \frac{1}{r_0}\right) \qquad \text{and} \qquad b = \frac{1}{2}\left|r_0 - \frac{1}{r_0}\right|$$

and the foci ± 1.

Conclusion: The Joukowski–function maps a collection of circles $r \equiv \mathrm{const.}$ onto a collection of kofocal ellipses. The two circles $r \equiv r_0$ and $r \equiv 1/r_0$ are mapped onto the same ellipse.

Geometric behaviour of the Joukowski–function.

For the image of the ray $\varphi \equiv \varphi_0$ we obtain

$$\left.\begin{array}{rcl} u &=& \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\varphi_0) \\ v &=& \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\varphi_0) \end{array}\right\} \qquad 0 < r < \infty,$$

and therefore for the positive x-axis $\varphi_0 = 0$

$$\left.\begin{array}{rcl} u &=& \frac{1}{2}\left(r + \frac{1}{r}\right) \\ v &=& 0 \end{array}\right\} \qquad 0 < r < \infty,$$

the subset $\{(u, 0) \,|\, 1 \le u < \infty\}$ of the u-axes.

In analogy we obtain for the negative x-axis $\varphi_0 = \pi$ the piece $-\infty < u < -1$.

The rays $\varphi_0 = \pi/2$ (positive y-axis) and $\varphi_0 = 3\pi/2$ (negative y-axis) are mapped onto the (complete) v-axis.

Geometric behaviour of the Joukowski–function.

If $\varphi_0 \notin \{0, \pi/2, \pi, 3\pi/2\}$ we can eliminate r. Thus we obtain the hyperbola

$$\frac{u^2}{\cos^2(\varphi_0)} - \frac{v^2}{\sin^2(\varphi_0)} = 1$$

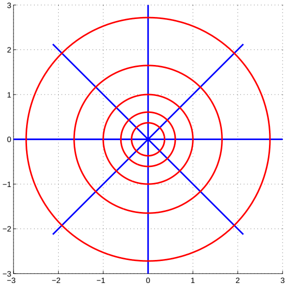
with the semiaxes

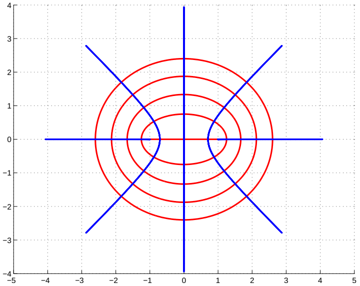
$$a = |\cos(\varphi_0)| \qquad \text{and} \qquad b = |\sin(\varphi_0)|.$$

The distance of the foci from the origin is

$$\sqrt{a^2 + b^2} = \sqrt{\cos^2(\varphi_0) + \sin^2(\varphi_0)} = 1.$$

Therefore the two foci are in ± 1.





Domain. **Image under the Joukowski–function.**

Additional remarks to the Joukowski–function.

1. The Joukowski–function maps the net of polar coordinates onto a net of ellipses and hyperbolas which intersect orthogonally. Thus the Joukowski–function is isogonal.

2. The Joukowski–function is not injective on its domain $\mathbb{C} \setminus \{0\}$ since for every $z \in \mathbb{C} \setminus \{\pm 1, 0\}$ it is $z \neq 1/z$, but $f(z) = f(1/z)$.

3. On the following two restrictions of the domain the Joukowski–function becomes injectiv.
   - On the complement of the unit circle $D(f) = \{z \in \mathbb{C} \mid |z| > 1\}$.
   - On the upper half plane $D(f) = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

4. The inverse function $w = f^{-1}(z)$ of the Joukowski–function $f(w)$ is obtained by solving the related quadratic equation

$$w^2 - 2zw + 1 = 0$$

w.r.t. w in the related domain $D(f)$, thus $w = z + \sqrt{z^2 - 1}$.

3.1 The stereographic projection

Preliminaries: In analysing rational functions

$$R(z) = \frac{p(z)}{q(z)} \qquad \text{with polynomials } p, q : \mathbb{C} \to \mathbb{C}$$

it is reasonable to close the gaps in the domain (i.e. the zero's of $q(z)$) by attributing to $R(z)$ in theese points the “value” ∞ if at such point not at the same time the nominator $p(z)$ vanishes.

Notation: If $z^* \in \mathbb{C}$ is a zero of q, i.e. $q(z^*) = 0$, and $p(z^*) \neq 0$, then $R(z^*) = \infty$, i.e. the codomain of R is enlarged by adding the "number" ∞.

Definition: In the extension $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ of the complex plane ∞ is denoted as infinitely far point.

Extension of the rules of calculus for $\mathbb{C}^*$.

In the extended complex plane $\mathbb{C}^*$ in addition to the usual rules in $\mathbb{C}$ we define the following rules.

$$\begin{aligned} a + \infty &:= \infty \quad \text{for } a \in \mathbb{C} \\ a \cdot \infty &:= \infty \quad \text{for } a \in \mathbb{C} \setminus \{0\} \\ a/\infty &:= 0 \quad \text{for } a \in \mathbb{C} \end{aligned}$$

Warning: The combintions $0 \cdot \infty$ and $\infty \pm \infty$ cannot be defined reasonably (i.e. without contradictions).

Topological meaning: The extended complex plane $\mathbb{C}^*$ is a topological space. For a complex sequence $\{z_n\}_n$, $z_n \neq 0$, we have

$$z_n \to \infty \quad \text{for } n \to \infty \qquad \Longleftrightarrow \qquad 1/z_n \to 0 \quad \text{for } n \to \infty$$

The space $\mathbb{C}^*$ is sequentially compact, i.e. every sequence in $\mathbb{C}^*$ as (at least) one limit point. Thus $\mathbb{C}^*$ is denoted as compactification of $\mathbb{C}$.

The stereographic projection.

Definition: The stereographic projection is the map $P : \mathbb{S}^2 \to \mathbb{C}^*$ which maps the Riemann sphere

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$$

on the extended complex plan $\mathbb{C}^*$, in particular it maps a point $x \in \mathbb{S}^2$, $x \neq N = (0,0,1)^T$, onto the point in the x_1–x_2–plane (considered to lie below the sphere) which lies on a straight line from the north pole N of the sphere through the point x on the sphere. And N is mapped to $P(N) := \infty$.

The stereographic projection has the following analytical representation

$$z = P(x) = \frac{x_1 + ix_2}{1 - x_3} \in \mathbb{C}^* \qquad \text{for } = (x_1, x_2, x_3)^T \in \mathbb{S}^2.$$

Remark:

1. The stereographic projection $P : \mathbb{S}^2 \to \mathbb{C}^*$ is bijective.
2. The inverse map P^{-1} of P is given by

$$x = P^{-1}(z) = \left(\frac{z + \overline{z}}{1 + z\overline{z}}, \frac{z - \overline{z}}{i(1 + z\overline{z})}, \frac{z\overline{z} - 1}{1 + z\overline{z}} \right)^T \in \mathbb{S}^2 \qquad \text{for } z \in \mathbb{C}^*.$$

The geometry of the stereographic projection.

By a sperical image U of a set $B \subset \mathbb{C}^*$ in the following we unterstand the (original) domain which under the stereographic projection is mapped on B, i.e. $P(U) = B$.

Theorem: The stereographic projection has the following properties.

a) The spherical image of a straight line in $\mathbb{C}^*$ is a circle on $\mathbb{S}^2$ containing N.

b) A circle on $\mathbb{S}^2$, passing through N, is mapped under the stereographic projection on a straight line in $\mathbb{C}^*$.

c) The spherical image of a circle in $\mathbb{C}$ is a circle in $\mathbb{S}^2$, NOT passing through N.

d) A circle on $\mathbb{S}^2$, NOT passing through N, is mapped under the stereographic projection on a circle in $\mathbb{C}$.

e) The stereographic projection is conformal.

3.2 Möbius–transforms

Definition: A rational map of the form

$$w = T(z) = \frac{az+b}{cz+d} \qquad \text{with} \quad ad \neq bc$$

is called Möbius–transform.

Remark: For the Möbius–transform $T : \mathbb{C}^* \to \mathbb{C}^*$ it holds:

1. Nominator and denominator have no common zero.
2. It is $T(-d/c) = \infty$ and $T(\infty) = a/c$.
3. The map $T(z)$ is bijective with inverse map $T^{-1} : \mathbb{C}^* \to \mathbb{C}^*$

$$T^{-1}(w) = \frac{dw-b}{-cw+a}.$$

4. Analogy to the inverse of a (2×2)–matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Composition of Möbius-transforms.

Theorem: The composition of two Möbius-transforms is again a Möbius-transform. More precisely

$$
\begin{aligned}
w = T_1(z) &= \frac{az+b}{cz+d} \qquad \text{for } ad \neq bc \\
u = (T_2 \circ T_1)(z) = T_2(w) &= \frac{\alpha w+\beta}{\gamma w+\delta} \qquad \text{for } \alpha\delta \neq \beta\gamma \\
&= \frac{Az+B}{Cz+D}
\end{aligned}
$$

The coefficients A, B, C and D can be obtained from the matrix product

$$
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

Due to $\det(AB) = \det A \cdot \det B$ we have

$$
AD - BC = (ad - bc) \cdot (\alpha\delta - \beta\gamma) \neq 0
$$

Conformality of Möbius-transforms.

Theorem: Möbius-transforms are conformal, i.e. (generalized) circles in $\mathbb{C}^*$ are mapped by Möbius-transforms in (generalized) circles.

Proof: Use an appropriate decomposition for $c \neq 0$

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c} \cdot \frac{1}{cz+d}$$

Now we set

$$\begin{aligned} w_1 &= cz+d \\ w_2 &= \frac{1}{w_1} \\ w_3 &= \frac{a}{c} - \frac{ad-bc}{c} \cdot w_2 \end{aligned}$$

The maps w_1 and w_3 are linear and thus conformal.

Continuation of the proof.

It remains to show:

The inversion $w = f(z) = 1/z$ is a conformal map.

We use the detour via the stereographic projection, i.e. instead of $z \to 1/z$ we consider the three of maps

$$z \to \mathrm{x} := P^{-1}(z) \to \tilde{\mathrm{x}} \to P(\tilde{\mathrm{x}}) = \frac{1}{z}$$

Then we have

$$\mathrm{x} = P^{-1}(z) = \left(\frac{z+\bar{z}}{z\bar{z}+1}, \frac{z-\bar{z}}{i(z\bar{z}+1)}, \frac{z\bar{z}-1}{z\bar{z}+1} \right)^T$$

and

$$\begin{aligned} \tilde{\mathrm{x}} &:= P^{-1}\left(\frac{1}{z}\right) \\ &= \left(\frac{\frac{1}{z}+\frac{1}{\bar{z}}}{\frac{1}{z}\frac{1}{\bar{z}}+1}, \frac{\frac{1}{z}-\frac{1}{\bar{z}}}{i(\frac{1}{z}\frac{1}{\bar{z}}+1)}, \frac{\frac{1}{z}\frac{1}{\bar{z}}-1}{\frac{1}{z}\frac{1}{\bar{z}}+1} \right)^T \end{aligned}$$

Completion of the proof.

A simplification gives

$$\begin{aligned}\tilde{\mathrm{x}} &= \left(\frac{z+\bar{z}}{z\bar{z}+1}, -\frac{z-\bar{z}}{i(z\bar{z}+1)}, -\frac{z\bar{z}-1}{z\bar{z}+1}\right) \\ &= (x_1, -x_2, -x_3)^T\end{aligned}$$

Thus we obtain a map $F : S^2 \to S^2$ with

$$F(\mathrm{x}) = (x_1, -x_2, -x_3)^T$$

This map is a rotation of the sphere arount the x_1–axis by 180° and apparently confromal.

Therefore we have proofed that the three maps

$$z \to \mathrm{x} := P^{-1}(z) \to \tilde{\mathrm{x}} \to P(\bar{\mathrm{x}}) = \frac{1}{z}$$

are conformal. With this the inversion $z \to 1/z$ is conformal.

Remarks on the Möbius-transform.

Remark: The Möbius-transform

$$w = T(z) = \frac{az + b}{cz + d} \qquad \text{with } ad \neq bc$$

has the follwoing properties.

- (Generalized) circles through the point $-d/c$ are mapped by T on straight lines in the w-plane.
- All straight lines in the z-plane are mapped by T on (generalized) circles in the w-plane containing the point a/c.
- Circles not containing the point $-d/c$ are mapped by T on circles not containing the point a/c.

Theorem: Let $z_1, z_2, z_3 \in \mathbb{C}^*$ and $w_1, w_2, w_3 \in \mathbb{C}^*$ be pairwise different. Then there exists exactely one Möbius-transform $w = T(z)$ satisfying the interpolations

$$w_j = T(z_j) \qquad \text{für}\, j = 1, 2, 3.$$

The interpolating Möbius-transform $T(z)$ is given by the three-point-formula

$$\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

Definition: The expression

$$D(z_0, z_1, z_2, z_3) = \frac{z_0 - z_1}{z_0 - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

is called cross-ratio of the points z_0, z_1, z_2, z_3.

Example.

We are looking for the Möbius–transform with interpolation properties

z_i	1	i	0
w_i	i	$-i$	0

We obtain a unique Möbius–transform using the Ansatz

$$\frac{w-i}{w+i} : \frac{0-i}{0+i} = \frac{z-1}{z-i} : \frac{0-1}{0-i}$$

A simplification gives

$$-\frac{w-i}{w+i} = i\,\frac{z-1}{z-i}$$

or

$$(z-i)(w-i) = -i\,(z-1)(w+i)$$

This finally leads to gives

$$w = \frac{(1+i)z}{(1+i)z-2i}$$

Symmetry w.r.t. the circle.

Definition:
Let C in $\mathbb{C}$ be circle with center $z_0 \in \mathbb{C}$ and radius R. Two points $z, z' \in \mathbb{C}$ are called symmetric w.r.t. the circle C, if

$$(z - z_0)\overline{(z' - z_0)} = R^2$$

The map $z \to z'$ is called circle inversion on C or plane inversion on C.

Graphical representation of the plane inversion in the slide!

Remarks:

- A point z with $|z - z_0| \leq R$ is symmetrric w.r.t. a point z' with $|z' - z_0| \geq R$.
- If $|z - z_0| = R$, then z is symmetric to itself, i.e. $z' = z$.
- The point $z = z_0$ is symmetric to $z' = \infty$.

Möbius-transforms a circle symmetries.

Definition: Two points z, z' are called symmetric with respect to a straight line in $\mathbb{C}$, if z' is obtained from z by reflection across a line.

Theorem:
Möbius-transforms conserve symmetries w.r.t. (generalized) circles.

More precisely:
If C is a (generalized) circle in $\mathbb{C}^*$ and if z and z' are symmetric w.r.t. C, then the images z, z' of a Möbius-transform are symmetric w.r.t the to the (generalized) circle in $\mathbb{C}^*$, which is the image of C.

Example: We look for a Möbius-transform $w = T(z)$, such that the circle $|z| = 2$ is mapped on the circle $|w + 1| = 1$ with $T(-2) = 0$ and $T(0) = i$.
A Möbius–transform is uniquely determined if the transformation is given for three points. But we only have

$$z_1 = -2,\ z_2 = 0 \quad \text{and} \quad w_1 = 0,\ w_2 = i$$

Therefore one point is missing!

Continuation of the example.

According to the last theorem Möbius–transforms conserve symmetries w.r.t. generalized circles.

$$z_2 = 0 \quad \Rightarrow \quad z_3 = \infty \quad \text{is symmetric to } z_2 \text{ w.r.t. the circle } |z| = 2$$

Thus w_3 is the point symmetric to $w_2 = i$ w.r.t the circle $|w+1| = 1$ and therefore given by the condition $(w_2 + 1)\overline{(w_3 + 1)} = 1$, i.e.

$$w_3 = \frac{1}{2}(-1 + i)$$

Application of the three point formula gives

$$\frac{w-0}{w-i} : \frac{w_3 - 0}{w_3 - i} = \frac{z+2}{z-0} : \frac{z_3+2}{z_3-0}$$

What happens to

$$\frac{z_3+2}{z_3-0}$$

as $z_3 \to \infty$?

Completion of the example.

What happens to

$$\frac{z_3+2}{z_3-0}$$

as $z_3 \to \infty$?

It is

$$\frac{z_3+2}{z_3-0} = \frac{1+\frac{2}{z_3}}{1+\frac{0}{z_3}} \to 1 \quad \text{for } z_3 \to \infty$$

We obtain

$$\left(\frac{w}{w-i}\right) : \left(\frac{\frac{1}{2}(-1+i)}{\frac{1}{2}(-1+i)-i}\right) = \left(\frac{z+2}{z}\right)$$

and solving w.r.t w gives

$$w = T(z) = -\frac{z+2}{(1+i)z+2i}$$

Example.

For $b > a > 0$ we consider the Möbius-transform

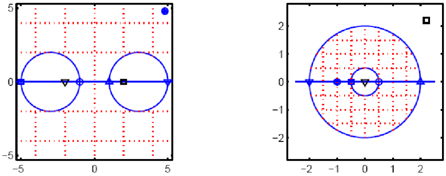
$$w = T(z) = \frac{z+p}{-z+p} \qquad \text{where } p = \sqrt{ab} \in (a,b)$$

Using T we obtain

$$\begin{aligned}
z_{1,2} &= \pm p & \rightarrow \quad & w_{1,2} = \infty, 0 \\
z_{3,4} &= a, b & \rightarrow \quad & w_{3,4} = \pm\frac{\sqrt{a}+\sqrt{b}}{\sqrt{b}-\sqrt{a}} = \pm\varrho \quad \text{with } |\varrho| > 1 \\
z_{5,6} &= -a, -b & \rightarrow \quad & w_{5,6} = \pm\frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}+\sqrt{b}} = \pm\frac{1}{\varrho} \\
z_{7,8} &= 0, \infty & \rightarrow \quad & z_{7,8} = 1, -1.
\end{aligned}$$

Continuation of the example.

- The x-axis is mapped by T onto the u-axis.
- Points which are symmetric with respect to the x-axis are mapped onto points which are symmetric w.r.t. the u-axis.
- Circles being symmetric w.r.t the x-axis are mapped onto circles beiing symmetric wi



Important applications: The electrostatic field in the exterior of two parallel conducting lines is mapped on the field of a cylindrical condensator.

4.1 Complex differentiation

Definition: Let $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ be a complex function. $f(z)$ is called complex differentiable in the point $z_0 \in D^0$ with derivative $f'(z_0)$, if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If $f(z)$ is complex differentiable in every point in the domain D, we call $f(z)$ holomorphic or analytic on D.

Remark:

1. The limit process $z \to z_0$ is intended in the complex plane, i.e. the approach $z \to z_0$ is arbitrary.
2. The division in the limit is a division in complex numbers.

4.1 Complex differentiation

Lemma: If $f(z)$ is real valued, i.e. $f : D \to \mathbb{R}$, $D \subset \mathbb{C}$ a domain, and if $f(z)$ is holomorphic on D, then $f(z)$ is a constant function.

Proof: We first consider the sequence $z_n \to z_0$ given by

$$z_n = z_0 + \frac{1}{n}$$

The the differential quotient is real for all $n \in \mathbb{N}$ since

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} = n(f(z_n) - f(z_0)) \in \mathbb{R}$$

On the other hand the sequence $z_n \to z_0$ with $z_n = z_0 + i/n$ gives a purely imanginary differential quotient

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} = \frac{n}{i}(f(z_n) - f(z_0)) \in \mathbb{C}$$

Since the function is holomorphic on D it follows

$$f'(z_0) = 0 \qquad \text{for all } z_0 \in D.$$

The Cauchy–Riemannschen equations.

Remark: If the function $f(z)$ is complex differentiable in z_0, then

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)-f'(z_0)(z-z_0)}{z-z_0} = 0$$

or equivalentely

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + o(|z-z_0|)$$

Let $f(z)$ be complex differentiable in z_0. We set

$$\gamma := f'(z_0),$$

then we obtain the equivalent formulation

$$f(z) = f(z_0) + \gamma(z-z_0) + \varepsilon(z)|z-z_0|$$

with $\varepsilon(z) \to 0$ as $z \to z_0$.

The Cauchy–Riemannschen equations.

We now use with $z = x + iy$ the formulation

$$f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$$

and

$$\gamma = \alpha + i\beta$$

Thus we obtain

$$\begin{aligned} u(z) &= u(z_0) + \alpha(x - x_0) - \beta(y - y_0) + \mathrm{Re}(\varepsilon(z)) \cdot |z - z_0| \\ v(z) &= v(z_0) + \beta(x - x_0) + \alpha(y - y_0) + \mathrm{Im}(\varepsilon(z)) \cdot |z - z_0| \end{aligned}$$

In matrix formulation this reads as

$$\begin{pmatrix} u(z) \\ v(z) \end{pmatrix} = \begin{pmatrix} u(z_0) \\ v(z_0) \end{pmatrix} + \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + \varepsilon(z) \cdot |z - z_0|$$

The Cauchy–Riemannschen equations.

We interpret $f(z)$ as vector valued, **totally differentiable** function of two variables, i.e.

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

with the Jacobian–matrix

$$Jf(x_0, y_0) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\Bigg|_{(x_0,y_0)} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

Theorem: The function $f(z)$ is complex differentiable in $z_0 \in D$ if and only if $f(z)$ as function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is totally differentiable and if the Cauchy–Riemannschen equations hold

$$\begin{aligned} u_x(z_0) &= v_y(z_0) \\ u_y(z_0) &= -v_x(z_0) \end{aligned}$$

Representation of the complex differentiation.

Corollary: If $f(z)$ is complex differentiable in $z_0 \in D$, then

$$f'(z_0) = u_x(z_0) + iv_x(z_0)$$

Proof: Since $f'(z_0) \in \mathbb{C}$ we can write

$$f'(z_0) = \tilde{u}(z_0) + i\tilde{v}(z_0)$$

From this we obtain

$$\begin{aligned} f'(z_0) \cdot (z - z_0) &= (\tilde{u}(z_0) + i\tilde{v}(z_0)) \cdot \Big[(x - x_0) + i(y - y_0)\Big] \\ &= \tilde{u} \cdot (x - x_0) - \tilde{v} \cdot (y - y_0) + i\left(\tilde{v} \cdot (x - x_0) + \tilde{u} \cdot (y - y_0)\right) \end{aligned}$$

Since f is totally differentiable in z_0 and since the Cauchy–Riemannschen equations are satisfied we have on the other side

$$\begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} u_x \cdot (x - x_0) - v_x(y - y_0) \\ v_x \cdot (x - x_0) + u_x(y - y_0) \end{pmatrix}$$

Holomorphic functions and the Laplace's equation.

Theorem: It $f \in \mathcal{C}^2$ is holomorphic on D, then

$$u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0,$$

i.e. both real and imaginary part of f satisfy the Laplace's equation.

Proof: If $f(z)$ is holomorphic, then

$$\begin{aligned}
\Delta u &= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \overset{C.R.}{=} \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0 \\
\Delta v &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \overset{C.R.}{=} -\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = 0
\end{aligned}$$

Also, the following inversion holds true: If $u = u(x, y)$ satisfies the Laplace's equation $\Delta u = 0$ on a connected domain, then there exists a differentiable function $v = v(x, y)$ such that $f(z) = u(z) + iv(z)$ on D is holomorphic.

Proof of the inversion.

Let $u = u(x, y)$ be given with $\Delta u = 0$. We are looking for a function $v = v(x, y)$, such that the Cauchy–Riemannschen equations are satisfied. Thus

$$v_x = -u_y \qquad v_y = u_x$$

From the C.R. equations it follows

$$\operatorname{grad} v = (v_x, v_y) = (-u_y, u_x) =: V = (V_1, V_2)$$

Therefore we are looking for a potential v with $\operatorname{grad} v = V$. If the integrability conditions

$$\frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial x} = 0$$

are satisfied, the existence of such a potential if guaranteed.

This is true since

$$\frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial x} = -u_{yy} - u_{xx} = -\Delta u = 0$$

Rules for the differentiation.

- The following rules hold:

$$\begin{aligned}(f \pm g)'(z_0) &= f'(z_0) \pm g'(z_0)\\ (f \cdot g)'(z_0) &= f'(z_0)g(z_0) + f(z_0)g'(z_0)\\ \left(\frac{f}{g}\right)'(z_0) &= \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}\end{aligned}$$

- Chain rule: If $f(z)$ is differentiable in z_0 and if $g(w)$ is differentiable in $w_0 = f(z_0)$, then

$$(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0)$$

- Derivation of the inverse function: If $f(z)$ is holomorphic and if $f'(z_0) \neq 0$, then $f(z_0)$ is locally bijective around z_0 and we have

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}, \quad w_0 = f(z_0)$$

The modified chain rule.

Lemma: If $f(z)$ is holomorphic on D and if $c : [a, b] \to D$ is a $\mathcal{C}^1$–curve in D, then

$$\frac{d}{dt} f(c(t)) = f'(c(t)) \cdot \dot{c}(t)$$

Proof: We have

$$\begin{aligned} \frac{d}{dt} f(c(t)) &= \frac{d}{dt} u(c(t)) + i \, \frac{d}{dt} v(c(t)) \\ &= (u_x \dot{c}_1 + u_y \dot{c}_2) + i \, (v_x \dot{c}_1 + v_y \dot{c}_2) \end{aligned}$$

In addition we have

$$\begin{aligned} f'(c(t)) \cdot \dot{c}(t) &= (u_x + i \, v_x) \cdot (\dot{c}_1 + i \, \dot{c}_2) \\ &= (u_x \dot{c}_1 - v_x \dot{c}_2) + i \, (v_x \dot{c}_1 + u_x \dot{c}_2) \end{aligned}$$

Both expressions are identical due to the C.R. equations.

Examples.

Example 1:

For $f(z) = z$ we obtain due to $u(x, y) = x$ and $v(x, y) = y$

$$f'(z) = u_x(z) + i\, v_x(z) = 1$$

Thus complex polynomials on $\mathbb{C}$ are holomorphic with

$$\frac{d}{dz}\left(\sum_{k=0}^{n} a_k z^k\right) = \sum_{k=1}^{n} a_k k z^{k-1}$$

Explicit calculation for $f(z) = z^2$: with

$$f(z) = z^2 = (x^2 - y^2) + i\, 2xy$$

we calculate

$$f'(z) = u_x(z) + i\, v_x(z) = 2x + i\, 2y = 2z$$

Examples.

Example 2: Rational functions, i.e. functions of the form

$$f(z) = \frac{p(z)}{q(z)}, \qquad p, q \text{ complex polynomials}$$

are complex differentiable at every point with $q(z) \neq 0$.

Example 3: The exponential function $f(z) = e^z = e^x(\cos y + i \sin y)$ is complex differentiable with $f'(z) = e^z$, since with

$$u(x, y) = e^x \cos y, \qquad v(x, y) = e^x \sin y$$

the C.R. equations are satisfied

$$u_x = v_y = e^x \cos y, \qquad u_y = -v_x = -e^x \sin y$$

and we have

$$f'(z) = u_x + i\, v_x = e^x \cos y + i\, e^x \sin y = e^z$$

More examples.

Example 4: The trigonometrc functions

$$\sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \qquad \cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

are according to example 3 holomorphic on $\mathbb{C}$ and we have the formulas for the derivatives in analogy to the real valued functions.

Example 5: Functions defined as complex power series,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

are holomorphic on the domain of convergence $K_r(z_0)$ with

$$f'(z) = \sum_{k=1}^{\infty} a_k k (z - z_0)^{k-1}$$

and thus on $K_r(z_0)$ at the same time arbitrary many times complex differentiable.

4.2 Conformal mappings

Theorem: Let $f : D \to \mathbb{C}$ be a holomorphic function on the domain $D \subset \mathbb{C}$ with $f'(z) \neq 0$ for all $z \in D$. Then locally in a point $z_0 \in D$ we have:

a) Angles between curves which intersect in z_0 are conserved under the transformation $w = f(z)$, including the rotational direction,

b) the expression $|f'(z_0)|$ is for all directions "leaving" z_0 the common scaling. In particular relations of lenghtes are conserved.

Mappings with these properties are called conformal mappings.

For conformal mappings we have the following inversion of the theorem.

Theorem: If $w = f(z)$ is a conformal mapping and if the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable, then $f(z)$ is complex differentiable and we have $f'(z) \neq 0$.

Proof of the first theorem.

Let c and d be two curves which at $t = 0$ go through z_0. The two tangential vectors in this point are $\dot{c}(0)$ and $\dot{d}(0)$ and for the angle γ between the tangential vectors we have

$$\gamma = \measuredangle\left(\dot{c}(0), \dot{d}(0)\right) = \arg\left(\dot{d}(0)\right) - \arg\left(\dot{c}(0)\right)$$

With f we obtain the two curves $f \circ c$ and $f \circ d$ in the codomain.
Th angle $\tilde{\gamma}$ between these two curves in $f(z_0)$ in the codomain is

$$\begin{aligned}
\tilde{\gamma} &= \measuredangle\left(f'(z_0)\dot{c}(0), f'(z_0)\dot{d}(0)\right) \\
&= \arg\left(f'(z_0)\dot{d}(0)\right) - \arg\left(f'(z_0)\dot{c}(0)\right) \\
&= \arg\left(f'(z_0)\right) + \arg\left(\dot{d}(0)\right) - \arg\left(f'(z_0)\right) - \arg\left(\dot{c}(0)\right) = \gamma
\end{aligned}$$

and w.r.t the scaling of lenghtes we calculate

$$\|\frac{d}{dt}\left(f \circ c\right)\| = |f'(z_0)\dot{c}(0)| = |f'(z_0)| \cdot |\dot{c}(0)|$$

Conformal transformations.

Definition: Let $f : D \to D'$ be a bijective and conformal mapping bewteen the domains $D \subset \mathbb{C}$ and $D' \subset \mathbb{C}$. Let $\Phi : D \to \mathbb{R}$ be a real valued twice continuously differentiable function on D. Then we call the function $\Psi : D' \to \mathbb{R}$ defined by

$$\Psi = \Phi \circ f^{-1}$$

the conformal transformation of Φ with mapping f.

Physical Applications: If $\Phi(z)$ is an unknown potential defined in the in the physical plane D, then Ψ is the related function in the modell plane D'.
In the following Φ and Ψ are potentials, i.e.

- electrostatic potentials;
- fluid dynamic potentials;
- temperature fields etc.

The vectors (Φ_x, Φ_y) and (Ψ_u, Ψ_v) are of particular interest.

The complex gradient.

Definition: For a real valued function $\Phi : D \to \mathbb{R}$ on a domain $D \subset \mathbb{C}$ we call with $z = x + iy$ the expression

$$\operatorname{grad} \Phi(z) = \frac{\partial \Phi}{\partial x} + i \, \frac{\partial \Phi}{\partial y}$$

the complex gradient of $\Phi(z)$.

Theorem: Let Ψ be the conformal transformation of Φ with mapping f. Then the two relations

$$\begin{aligned} \operatorname{grad}_z \Phi(z) &= \operatorname{grad}_w \Psi(f(z)) \cdot \overline{f'(z)} \\ \Delta_z \Phi(z) &= \Delta_w \Psi(f(z)) \cdot |f'(z)|^2 \end{aligned}$$

hold. **Proof:** By definition the conformal transformation of Φ with mapping f is given by

$$\Psi = \Phi \circ f^{-1}$$

Continuation of the proof.

We conclude $\Phi = \Psi \circ f$ and with $f(x,y) = u(x,y) + i\,v(x,y)$

$$\Phi(x,y) = \Psi(u(x,y), v(x,y))$$

We calculate

$$\begin{aligned}
\Phi_x &= \Psi_u u_x + \Psi_v v_x \\
\Phi_y &= \Psi_u u_y + \Psi_v v_y
\end{aligned}$$

For the complex gradient gwe have with $f'(z) = u_x + iv_x$

$$\begin{aligned}
\operatorname{grad}\Phi(z) &= (\Psi_u u_x + \Psi_v v_x) + i\,(\Psi_u u_y + \Psi_v v_y) \\
&= \Psi_u(u_x + i\,u_y) + \Psi_v(v_x + i\,v_y) \\
&\overset{C.R.}{=} \Psi_u(u_x - i\,v_x) + i\,\Psi_v(u_x - i\,v_x) \\
&= \operatorname{grad}\Psi(f(z)) \cdot \overline{f'(z)}
\end{aligned}$$

Completion of the proof.

Calculating the second derivative gives

$$\Phi_{xx} = \Psi_{uu} u_x^2 + 2\Psi_{uv} u_x v_x + \Psi_{vv} v_x^2 + \Psi_u u_{xx} + \Psi_v v_{xx}$$

$$\Phi_{yy} = \Psi_{uu} u_y^2 + 2\Psi_{uv} u_y v_y + \Psi_{vv} v_y^2 + \Psi_u u_{yy} + \Psi_v v_{yy}$$

Thus

$$\begin{aligned}\Delta\Phi &= \Psi_{uu}(u_x^2 + u_y^2) + 2\Psi_{uv}(u_x v_x + u_y v_y) \\ &\quad + \Psi_{vv}(v_x^2 + v_y^2) + \Psi_u \Delta u + \Psi_v \Delta v\end{aligned}$$

We use again the C.R. equations and obtain

$$\begin{aligned} u_x^2 + u_y^2 &= v_x^2 + v_y^2 = u_x^2 + v_x^2 = |f'(z)|^2 \\ u_x v_x + u_y v_y &= 0 \\ \Delta u &= \Delta v = 0 \end{aligned}$$

and therefore the desired result

$$\Delta\Phi = \Delta\Psi \cdot |f'(z)|^2$$

Practical applications of conformal transformations.

Corollary: Conformal transformations transform harmonic functions into harmonic functions.

Applications of conformal transformations: Lets consider the Dirichlet–problem for the Laplace equation, i.e. the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$

where $D \subset \mathbb{R}^2$ is a "complicated" two-dimensional domain.

With an appropriate conformal transformation we can solve the problem explicitely.

1. identify a conformal transformation which maps the physical domain D on a "simple" model domain D';
2. transform the boundary conditions on ∂D to boundary conditions on $\partial D'$ and solve the Dirichlet–problem on D';
3. Transform the solution back on the physical domain D.

An application: plain potential flow.

We would like to determine the velocity field of a stationary, curl– and source–free flow around a cylinder. Let $\mathrm{w} : \mathbb{R}^2 \to \mathbb{R}^2$ be the velocity field to be determined.

Then we have the equations

$$\operatorname{rot} \mathrm{w} = \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} = 0$$

$$\operatorname{div} \mathrm{w} = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0$$

If $D \subset \mathbb{R}^2$ is simply connected we obtain from the first condition

there exists a function $u : D \to \mathbb{R}$ with $\nabla u = -\mathrm{w}$

and from the second condition

there exists a function $v : D \to \mathbb{R}$ with $\nabla v = (w_2, -w_1)^T$

The complex flow potential.

We call

- the function u the velocity potential;
- the function v the stream function.

Related to the stream function we have stream lines which are solutions of the ordinary differential equations $y'(x) = w_2/w_1$ and given by

$$v(x, y) = \text{const.}$$

Definition: The complex funktion $\Phi = \Phi(x, y)$ defined by

$$\Phi(x, y) = u(x, y) + i\, v(x, y)$$

is called complex flow potential.

The complex flow potential $\Phi(z)$ is a holomorphic function, since we have the Cauchy–Riemann equations

$$\begin{aligned} u_x - v_y &= -w_1 - (-w_1) = 0 \\ u_y + v_x &= -w_2 + w_2 = 0 \end{aligned}$$

Continuation: plain potential flow.

The velocity field w can be calculated directly: due to

$$\Phi'(z) = u_x + i\,v_x = -w_1 + i\,w_2$$

it follows

$$\mathrm{w} = w_1 + i\,w_2 = -\overline{\Phi'(z)}$$

Our physical domain is diven by $D = \{z \in \mathbb{C} : |z| > R\}$ and the related model domain is

$$D' = \{z \in \mathbb{C} \mid \operatorname{Im} z \neq 0 \,\text{und}\, |\operatorname{Re} z| > 1\}$$

The Joukowski–function $f(z)$ given by

$$f(z) = \frac{1}{2}\left(\frac{z}{R} + \frac{R}{z}\right)$$

is a conformal transformation from D on D'.

Continuation: plain potential flow.

In the model plane we can assume a homogeneous velocity field, i.e. in D' we have

$$\mathrm{W} = \text{const.} = (V_\infty, 0)^T$$

since a infinitely flat plate is not interacting with a given homogeneous flow in the direction of the real axis with velocity V_∞.

For the velocity potential $U(W)$ we have the equation

$$\text{grad}\, U(W) = -(V_\infty, 0)^T$$

and from this follows

$$U(w) = -V_\infty\, W_1$$

Also there is a stream function $V(W)$

$$\text{grad}\, V(W) = (0, -V_\infty)^T \qquad \Rightarrow \qquad V(w) = -V_\infty\, W_2$$

Continuation: plain potential flow.

In the physical plane we can assume that

$$\lim_{z\to\infty} \operatorname{grad} \Phi(z) = -v_\infty$$

i.e. at infinity the undisturbed flow does not "feel" any obstacle.

Because of the relation

$$\operatorname{grad} \Phi(z) = \operatorname{grad} \Psi(f(z)) \cdot \overline{f'(z)}$$

it follows with

$$f'(z) = \frac{1}{2}\left(\frac{1}{R} - \frac{R}{z^2}\right)$$

the relation $V_\infty = 2Rv_\infty$.

For the complex flow potential we have

$$\Psi(W) = -2Rv_\infty(\operatorname{Re} W + i \operatorname{Im} W)$$

Continuation: plain potential flow.

Now we consider the back–transformation in the physical plane, i.e.

$$\Phi(z) = (\Psi \circ f)(z) = -2Rv_\infty(\mathrm{Re}\, f(z) + i\, \mathrm{Im}\, f(z))$$

For the Joukowski–function

$$f(z) = \frac{1}{2}\left(\frac{z}{R} + \frac{R}{z}\right)$$

it is

$$\mathrm{Re}\, f(z) = \frac{1}{2}\left(\frac{x}{R} + \frac{Rx}{x^2+y^2}\right) \qquad \mathrm{Im}\, f(z) = \frac{1}{2}\left(\frac{y}{R} - \frac{Ry}{x^2+y^2}\right)$$

With this in the physical plane we obtain the velocity potential $u(z)$

$$u(z) = u(x,y) = -v_\infty\left(x + \frac{R^2 x}{x^2+y^2}\right)$$

Continuation: plain potential flow.

We obtain for the stream function

$$v(z) = v(x, y) = -v_\infty \left(y - \frac{R^2 y}{x^2 + y^2} \right)$$

The velocity field w around the cylinder is given by

$$\mathrm{w} = -\nabla u = -v_\infty \left(\frac{(x^2+y^2)^2 - R^2(x^2-y^2)}{(x^2+y^2)^2}, -\frac{2R^2 xy}{(x^2+y^2)^2} \right)$$

In particular we have:

- In the two points $(-R, 0)$ and $(R, 0)$ the velocity is zero,

$$\mathrm{w}(-R, 0) = \mathrm{w}(R, 0) = (0, 0)^T$$

- The velocity is maximal in the two points $(0, -R)$ and $(0, R)$ with

$$w_{\mathrm{max}} = 2v_\infty$$

5.1 Examples for complex integration

Definition: A complex valued function $f : [a, b] \to \mathbb{C}$ of a real variable is integrable, if real– and imaginary part of f are integrable, and we have:

$$\int_a^b f(t)\,dt := \int_a^b \mathrm{Re}(f(t))\,dt + i \int_a^b \mathrm{Im}(f(t))\,dt = Re^{i\varphi}$$

The following properties in analogy to the intergration in the real numbers are valid Linearity. In addition we have

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$$

Proof: We calculate

$$\begin{aligned}\left|\int_a^b f(t)\,dt\right| &= R = e^{-i\varphi} \int_a^b f(t)\,dt = \int_a^b e^{-i\varphi} f(t)\,dt = \int_a^b \mathrm{Re}(e^{-i\varphi} f(t))\,dt \\ &\leq \int_a^b |e^{-i\varphi} f(t)|\,dt = \int_a^b |f(t)|\,dt\end{aligned}$$

Complex integration in analogy to curve integrals.

Real analysis: Let $c : [a, b] \to D \subset \mathbb{R}^n$ a piecewise $\mathcal{C}^1$–curve, $f : D \to \mathbb{R}$ and $\mathrm{F} : D \to \mathbb{R}^n$ are given. Then we have defined in Analysis II and III the line integrals of scalar and vector fileds

$$\int_c f(\mathrm{x}) ds := \int_a^b f(c(t)) \|\dot{c}\| \, dt$$

or

$$\int_c \mathrm{F}(\mathrm{x}) \, d\mathrm{x} := \int_a^b \langle \mathrm{F}(c(t)), \dot{c}(t) \rangle \, dt$$

Definition: Let $D \subset \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ continuous and $c : [a, b] \to D$ a piecewise $\mathcal{C}^1$–curve. Then

$$\int_c f(z) \, dz := \int_a^b f(c(t)) \dot{c}(t) \, dt$$

is the complex integral of $f(z)$ along the curve c.

Properties of the complex integral.

- The value of the compelx integral is independent of the parameterisation of the curve.
- Changing the orientation we have

$$\int_{-c} f(z)\,dz = -\int_c f(z)\,dz$$

We denote $(-c)(t) := c(b + t(a-b))$, $0 \le t \le 1$.

- Linearity

$$\int_c (\alpha f(z) + \beta g(z))\,dz = \alpha \int_c f(z)\,dz + \beta \int_c g(z)\,dz \qquad \text{für } \alpha, \beta \in \mathbb{C}$$

- Additivity with respect to the path of integration:

$$\int_{c_1+c_2} f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$$

Additional properties of the complex integral

We have the estimate

$$\left|\int_c f(z)\,dz\right| \le \sup_{z\in image(c)} |f(z)| \cdot \underbrace{\int_a^b |\dot{c}(t)|\,dt}_{lenghtofthepath \;\; L(c)}$$

Proof We calculate directly

$$\begin{aligned}
\left|\int_c f(z)\,dz\right| &= \left|\int_a^b f(c(t))\dot{c}(t)\,dt\right| \\
&\le \int_a^b |f(c(t))|\,|\dot{c}(t)|\,dt \\
&\le \sup_{a\le t\le b} |f(c(t))| \cdot \int_a^b |\dot{c}(t)|\,dt
\end{aligned}$$

Example 1:

Let $f(z) = z$ and $c(t) = re^{it}$ with $0 \le t \le 2\pi$. Then we have

$$\begin{aligned}
\oint_c z\,dz &= \int_0^{2\pi} re^{it} \cdot \left(rie^{it}\right) dt \\
&= ir^2 \int_0^{2\pi} e^{2it}\,dt \\
&= ir^2 \int_0^{2\pi} (\cos(2t) + i\,\sin(2t))\,dt \\
&= -r^2 \int_0^{2\pi} \sin(2t))\,dt + i\,r^2 \int_0^{2\pi} \cos(2t))\,dt \\
&= 0
\end{aligned}$$

Additional examples of complex integration.

Example 2:

Let $f(z) = \bar{z}$ and $c(t) = re^{it}$ with $0 \leq t \leq 2\pi$. then it is

$$\oint_c \bar{z}\, dz = \int_0^{2\pi} re^{-it} \cdot \left(rie^{it}\right) dt = ir^2 \int_0^{2\pi} dt = r^2 \cdot 2\pi i$$

Example 3:

Let $f(z) = 1/z$ and $c(t) = re^{it}$ with $0 \leq t \leq 2\pi$. Then it is

$$\oint_c \frac{1}{z}\, dz = \oint_c \frac{\bar{z}}{|z|^2}\, dz = \frac{1}{r^2} \oint_c \bar{z}\, dz = 2\pi i$$

Example 4: With $c(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ we have the relation

$$\oint_c (z - z_0)^n\, dz = \begin{cases} 2\pi i & : \quad \text{for } n = -1 \\ 0 & : \quad \text{for } n \in \mathbb{Z} \setminus \{-1\} \end{cases}$$

Example 4:

$$
\begin{aligned}
\oint_c (z - z_0)^n \, dz &= \int_0^{2\pi} \left(re^{it}\right)^n \cdot (rie^{it}) \, dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt \\
&= r^{n+1} \left(- \int_0^{2\pi} \sin((n+1)t)) \, dt + i \int_0^{2\pi} \cos((n+1)t)) \, dt \right) \\
&= \begin{cases} 2\pi i & : \quad \text{für } n = -1 \\ 0 & : \quad \text{for } n \in \mathbb{Z} \setminus \{-1\} \end{cases}
\end{aligned}
$$

Only for $n = -1$ the integral is not vanishing and we have

$$\oint_c \frac{1}{z - z_0} \, dz = 2\pi i$$

Question: Why this?

Uniform convergence and complex integration.

Theorem: Let $f(z) = \sum\limits_{k=0}^{\infty} f_k(z)$ be a series of continuous functions, which on a domain $D \subset \mathbb{C}$ converges uniformly. Let $c : [a, b] \to D$ be a piecewise $\mathcal{C}^1$–curve, then

$$\int_c f(z)\,dz = \sum_{k=0}^{\infty} \int_c f_k(z)\,dz$$

Proof: SInce the series of continuous functions converges uniformly also the limit function $f(z)$ is continuous and thus integrable

$$\int_c f(z)\,dz - \sum_{k=0}^{n} \int_c f_k(z)\,dz = \int_c R_n(z)\,dz$$

Uniform convergence means

$$\forall\, \varepsilon > 0 \;:\; \exists\, N(\varepsilon) \;:\; \forall\, n \geq N,\, z \in D \;:\; |R_n(z)| < \varepsilon$$

Continuation of the proof.

From the uniform convergence we conclude

$$\left| \int_c R_n(z)\, dz \right| \leq \varepsilon \cdot L(c)$$

and thus

$$\lim_{n\to\infty} \int_c R_n(z)\, dz = 0$$

Example: Let

$$c(t) = re^{it} \qquad \text{with } 0 \leq t \leq 2\pi$$

and $|z_0| > r$. Then:

$$\oint_{|z|=r} \frac{dz}{z - z_0}\, dz = 0$$

Note: The point z_0 lies outside the circle $c(t)$.

Continuation of the example.

We calculate directly using the geometric series

$$\oint_{|z|=r} \frac{dz}{z-z_0} = -\frac{1}{z_0} \oint_{|z|=r} \frac{dz}{1-\frac{z}{z_0}} = -\frac{1}{z_0} \oint_{|z|=r} \sum_{k=0}^{\infty} \frac{1}{z_0^k} z^k \, dz$$

since it is

$$\left| \frac{z}{z_0} \right| < 1$$

Due to the uniform convergence it is

$$\frac{1}{z_0} \oint_{|z|=r} \sum_{k=0}^{\infty} \frac{1}{z_0^k} z^k \, dz = \sum_{k=0}^{\infty} \frac{1}{z_0^{k+1}} \oint_{|z|=r} z^k \, dz = 0$$

since we can exchange integration and summation.

Anticipation of the Laurent–series.

Example: A series of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \underbrace{\sum_{k=0}^{\infty} a_k (z - z_0)^k}_{\text{in analogy to the Taylor–series}} + \underbrace{\sum_{k=-\infty}^{-1} a_k (z - z_0)^k}_{\text{negativ powers}}$$

is called a Laurent–serie.

It is comverging locally uniformly and absolutely in the ring

$$0 \leq R_1 < |z - z_0| < R_2$$

For $R_1 < r < R_2$ and $c(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ we have

$$\oint_{|z-z_0|=r} f(z)\, dz = \sum_{k=-\infty}^{\infty} a_k \oint_{|z-z_0|=r} (z - z_0)^k\, dz = 2\pi\, i\, a_{-1}$$