Mathematics Department
Prof. Dr. I. Gasser

WiSe 2022/23

Exam Complex functions

06. March 2022

Please mark each page with your name and your matriculation number.

Please write your surname, first name and matriculation number in block letters each into the following designated fields. These entries will be stored on data carriers.

Surname:																
First name:																
Matr.-No.:																

BP:	AIW	BU	BV	CI CS	ET	EUT	GES	IN IIW	LUM	MB	MTB MEC	SB	VT	

I have been instructed about the fact that the required test performance will only be assessed if the TUHH examination office can assure my official admission before the exam's beginning.

(Signature)

Task no.	Points	Evaluater
1		
2		
3		
4		

$\sum =$

Task 1) [4 points]

Let i be the imaginary unit. Determine all complex solutions of the following equation

$$\left(e^{i\frac{\pi}{8}} \cdot z\right)^4 = -16i\,.$$

Provide a sketch of their positions in the complex plane.

Solution for task 1)

With $z = re^{i\phi}$ one obtainds

$w := \left(e^{i\frac{\pi}{8}} \cdot z\right)^4 = e^{i\frac{\pi}{2}} \cdot z^4 = e^{i\frac{\pi}{2}} \cdot r^4 \cdot e^{i4\phi} \overset{!}{=} 16e^{-i\frac{\pi}{2}}.$ [**Ansatz: 1 point**]

$|w| = r^4 \overset{!}{=} 16 \iff r = 2\,.$ [**1 point**]

$e^{i4\phi} = e^{-i\pi} \iff 4\phi = -\pi + 2k\pi$
$\iff \phi = \frac{-\pi}{4} + \frac{k\pi}{2}, \qquad k = -1, 0, 1, 2.$ [**1 point**]

Sketch: [**1 point**]

Task 2) [4 points]

Let i be the imaginary unit, $z = x + iy$, $x, y \in \mathbb{R}$ and let u denote the function

$$u : \mathbb{R}^2 \to \mathbb{R}, \qquad u(x, y) = 4x^2 - 4y^2 + 2e^{3x} \sin(3y).$$

a) Show that the function u is harmonic.

b) Determine all conjugate harmonic functions v to u, that is, all functions v for which $f = u + iv$ is complex differentiable everywhere in $\mathbb{C}$.

Solution for 2:

a) i) $u_{xx} = (8x + 6e^{3x} \sin(3y))_x = 8 + 18e^{3x} \sin(3y)$.

$u_{yy} = (-8y + 6e^{3x} \cos(3y))_y = -8 - 18e^{3x} \sin(3y)$.

So $\Delta u = u_{xx} + u_{yy} = 0$.

ii) $f(z) = u(z) + iv(z)$ with

$u(x + iy) = \mathrm{Re}\,(f(x + iy)) = 4x^2 - 4y^2 + 2e^{3x} \sin(3y)$.

$v_y = u_x = 8x + 6e^{3x} \sin(3y) \iff v(x, y) = 8xy - 2e^{3x} \cos(3y) + c(x)$,

$-u_y = 8y - 6e^{3x} \cos(3y) \overset{!}{=} v_x = 8y - 6e^{3x} \cos(3y) + c'(x)$

$\iff c'(x) = 0 \implies v(x, y) = 8xy - 2e^{3x} \cos(3y) + C, \qquad C \in \mathbb{R}$.

Given $\quad f(z) \;=\; \dfrac{1}{(z-2)^2(z+1)}\,,$

a) determine and classify all isolated singularties of f.

b) calculate the residues of all isolated singularities of f.

c) provide the complex partial fraction representation of f.

d) find the number of different Laurent expansions for f about $z_0 = 2$.

e) determine the Laurent expansion for f about $z_0 = 2$ which converges to $f(-2)$ at the point $z^* = -2$.

Solution for task 3) $\quad f(z) \;=\; \dfrac{1}{(z-2)^2(z+1)}\,.$

a) Roots of the denominator: $z_1 = -1$ and $z_2 = 2$.
There is a simple pole in z_1 and a pole of order 2 in z_2 . **[1 point]**

b) Residues [**2 points**]

$$Res\, f(-1) = \left[\frac{1}{(z-2)^2}\right]_{z=-1} = \frac{1}{9}\,.$$

$$Res\, f(2) = \left[\left(\frac{1}{z+1}\right)'\right]_{z=2} = \left[\left(\frac{-1}{(z+1)^2}\right)'\right]_{z=2} = -\frac{1}{9}\,.$$

c) $f(z) = h_f(z;2) + h_f(z;-1)$:

$$h_f(z;-1) = \frac{Res(f;-1)}{z+1} = \frac{1}{9(z+1)} \; \textbf{[1 point]}$$

$$f(z) = \tfrac{1}{(z-2)^2} \cdot \underbrace{\frac{1}{z+1}}_{g(z)} = \tfrac{1}{(z-2)^2}\left(g(2) + g'(2)(z-2) + \cdots\right. . \; \textbf{[1 point]}$$

$$\Longrightarrow h_f(z;2) = \tfrac{1}{3(z-2)^2} - \tfrac{1}{9(z-2)}$$

So: $\quad f(z) = \tfrac{1}{9(z+1)} + \tfrac{1}{3(z-2)^2} - \tfrac{1}{9(z-2)} \qquad$ **(1 point)**

d) Two series. One for $0 < |z-2| < 3$ and another for $|z-2| > 3$. . **[1 point]**

e) Since $|-2-z_0| = |-2-2| = 4 > 3$ we want to determine the Laurent series for $|z-2| > 3$ (approximated at $z_0 = 2$). In this ring it holds that

$$f(z) = \frac{1}{(z-2)^2} \cdot \underbrace{\frac{1}{z+1}} = \frac{1}{(z-2)^2} \cdot \frac{1}{(z-2)+3}$$

$$= \frac{1}{(z-2)^2} \cdot \frac{1}{z-2} \cdot \frac{1}{1-(-\frac{3}{z-2})} = \frac{1}{(z-2)^3} \sum_{k=0}^{\infty} \frac{(-3)^k}{(z-2)^k}$$

$$= \sum_{k=0}^{\infty} \frac{(-3)^k}{(z-2)^{k+3}} = \sum_{k=-\infty}^{-3} (-3)^{-k-3}(z-2)^k$$

[3 points]

Task 4: (2 points)

Calculate $\int_{-\infty}^{\infty} \frac{1}{(x^2+25)(x^2+4)}\, dx.$

Solution:

$f(z) := \frac{1}{(z^2+25)(z^2+4)}$ has two singularities (simple poles) in the upper half plane ($z_1 = 2i$ and $z_2 = 5i$). It holds that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(x^2+25)(x^2+4)}\, dx &= 2\pi i (Res_f(5i) + Res_f(2i)) \\
&= 2\pi i \left(\left[\frac{1}{(z+5i)(z^2+4)} \right]_{z=5i} + \left[\frac{1}{(z+2i)(z^2+25)} \right]_{z=2i} \right) \\
&= 2\pi i \left(\left[\frac{1}{(10i)(-21)} \right] + \left[\frac{1}{(4i)(21)} \right] \right) \\
&= \frac{2\pi}{21} \left(-\frac{1}{(10)} + \frac{1}{4} \right) = \frac{3\pi}{210}
\end{aligned}$$