Mathematics Department
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SoSe 2022

Exam Complex functions

06. September 2022

Please mark each page with your name and your matriculation number.

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I was instructed about the fact that the required test performance will only be assessed if the TUHH examination office can assure my official admission before the exam's beginning.

(Signature)

Task no.	**Points**	**Evaluater**
1		
2		
3		

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Task 1) [5 points]

a) For which $k \in \mathbb{R}$ is the function

$$f : \mathbb{C} \to \mathbb{C}\,,\; f(z) := (\mathrm{Re}\,(z))^2 - (\mathrm{Im}\,(z))^2 + k \cdot \mathrm{Im}\,(z) + 2i \cdot \mathrm{Re}\,(z) \cdot [\mathrm{Im}\,(z) + 1]$$

complex differentiable in every point in $\mathbb{C}$?

b) In which points in $\mathbb{C}$ does the function

$$g : \mathbb{C} \to \mathbb{C}, \qquad g(z) := z \cdot e^z$$

preserve angles?

Solution for 1:

a) With the usual notation $z = x + iy$ it holds that

$f(z) := \underbrace{x^2 - y^2 + k \cdot y}_{u(x,y)} + i\,\underbrace{(2xy + 2x)}_{v(x,y)}$.

The Cauchy-Riemann equations are as follows:

$u_x = 2x \overset{!}{=} v_y = 2x$ so $\boxed{\text{arbitrary } k \in \mathbb{R}}$

and

$-u_y = 2y - k \overset{!}{=} v_x = 2y + 2$ so $\boxed{k = -2}$.

For $k = -2$, f is complex differentiable everywhere in $\mathbb{C}$. **(3 points)**

b) $g(z) := z \cdot e^z$

g is differentiable everwhere in $\mathbb{C}$ since z and e^z are differentiable everywhere in $\mathbb{C}$. It holds

$g'(z) = e^z + z \cdot e^z = (z+1)e^z$.

g preserves angles where $f'(z) \neq 0$, so for all $z \neq -1$.

(2 points)

Task 2) [7 points]

a) Determine a Möbius transform $T : \mathbb{C}^* \to \mathbb{C}^*, \quad T(z) := \dfrac{az+b}{cz+d}$ that satisfies

$$T(i) = 0\,, \qquad T(\infty) = 2\,, \qquad T(-1) = \infty\,.$$

b) Which generalized circles in $\mathbb{C}$ are mapped onto lines by T?

c) Determine the images of the following generalized circles of T from part a).
$K :=$ real axis,
$\tilde{K} := \{z \in \mathbb{C} : |z| = 1\}$.

Solution for 2) [7 points]

a)
$$T(i) = 0\,, \qquad T(-1) = \infty\,. \iff \qquad T(z) = \frac{a(z-i)}{z+1}.$$
$$T(\infty) = 2\,, \qquad \Longrightarrow \qquad T(z) = \frac{2z-2i}{z+1}. \qquad \textbf{[2 points]}$$

b) A generalized circle is mapped onto a line if and only if the point -1 is located on that generalized circle. [**1 point**]

c) $K = \mathbb{R}$ [**2 points**]
Since $-1 \in \mathbb{R}$, the image of the real axis is a line g_1.
Since $T(\infty) = 2$, 2 is also located on g_1.
We determine the image of another real number, for example
$T(0) = -2i$.
Consequently, $T(\mathbb{R})$ is the line that passes through 2 and $-2i$:
$T(\mathbb{R}) = g_1 = \{w = u + iv \in \mathbb{C} : v = u - 2\}$.

$\tilde{K} := \{z \in \mathbb{C} : |z| = 1$. Unit circle. [**2 points**]
Since $-1 \in \tilde{K}$, the image of the unit circle is a line g_2.
Since $T(i) = 0$, g_2 passes through the origin.
Since $\tilde{K}$ is symmetric to $\mathbb{R}$ (in its domain), g_2 is perpendicular to g_1, so
$g_2 = \{w = u + iv \in \mathbb{C} : v = -u\}$.
Alternatively, the image can be determined by taking another point on the unit circle, for example $T(1) = 1 - i$ which leads to the same result.

Task 3: (8 points)

Let $\Gamma := \{z(t) = 2i + 5 \cdot e^{it} \,|\, t \in [0, 2\pi]\}$ be the circle with radius 5 around $2i$ which is traversed once (positively)

Calculate the following path integrals.

a) $\displaystyle\int_\Gamma \frac{z^2}{z-6}\,dz$.

b) $\displaystyle\int_\Gamma \frac{z^2}{(z-2i)(z+i)}\,dz$.

c) $\displaystyle\int_\Gamma \frac{z^2}{(z+i)^2}\,dz$.

d) $\displaystyle\int_\Gamma \overline{(z-2i)}\,dz$, where $\overline{z}$ is the complex conjugate of z.

Solution for 3:

a) $\displaystyle\int_\Gamma \frac{z^2}{z-6}\,dz = 0$ (CIT) [**1 point**]

b) According to the residue theorem it holds that:

$$\int_\Gamma \frac{z^2}{(z+i)(z-2i)}\,dz = 2\pi i\left(\left[\frac{z^2}{z-2i}\right]_{z=-i} + \left[\frac{z^2}{(z+i)}\right]_{z=2i}\right)$$
$$= 2\pi i\left(\frac{-1}{-3i} + \frac{-4}{3i}\right) = 2\pi\left(\frac{1}{3} - \frac{4}{3}\right) = -2\pi\,. \textbf{ (3 points)}$$

c) According to the Cauchy integral formular for derivatives we obtain

$$\int_\Gamma \frac{z^2}{(z+i)^2}\,dz = \frac{2\pi i}{1!}\left[(z^2)'\right]_{z=-i} = 2\pi i \cdot 2(-i) = 4\pi\,. \textbf{ (2 points)}$$

d)

$$\int_\Gamma \overline{(z-2i)}\,dz = \int_0^{2\pi} \overline{2i + 5e^{it} - 2i} \cdot \dot{\Gamma}(t)dt = \int_0^{2\pi} \overline{5e^{it}} \cdot 5ie^{it}dt$$
$$= \int_0^{2\pi} 25i \cdot e^{-it} \cdot e^{it}dt = 2\pi \cdot 25i = 50\pi i. \textbf{ (2 points)}$$